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Abstract

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Keywords

electromagnetically induced transparency, lossless propagation, population transfer, pulses, continuum, symmetry

Disciplines

Physics

Matched Optical Solitary Waves for Three- and Five-Level Systems

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The problem of rendering a multilevel optical medium transparent has been a subject of considerable interest for many years [1]. Recently, Harris, Field, and Imamoglu [2] introduced an interesting concept of electromagnetically induced transparency (EIT), in which an approximate transparency for the pump probe laser pulse is obtained by applying a strong laser or electromagnetic field on the Stokes transition in a three-level lambda system.

In this Letter, we present *exact* analytic solutions for a pair of solitary waves which can propagate through a medium of three-level atoms and a medium of five-level atoms without loss and with their shapes invariant. The number of *free* parameters present in the solutions is surprisingly large. Some remarkable features of these solitary waves include the variety in their shapes, their different degree of sensitivity to small perturbations, and the variety of initial condition required for the atoms. The three- and five-level atoms are assumed chainwise dipole connected, i.e., for levels numbered 1 to N, where $N = 3$ or 5; 1 is connected to 2, 2 is connected to 3, etc. For $N = 3$, the level configuration can be a Λ (a V) configuration where level two is higher (lower) in energy than levels one and three. For $N = 5$, the level configuration can be a M (a W) configuration where level two is higher (lower) than levels one and three, and level four is higher (lower) than levels three and five. We shall present our solutions in one compact form for both the $N = 3$ and $N = 5$ systems and for both types of level configuration Λ or M and V or W. To do so, we introduce a parameter n, where $n = 1$ for $N = 3$, and $n = 2$ for $N = 5$ [i.e., $n = \frac{1}{2}(N-1)$], and a parameter ε, where $\varepsilon = +1$ for Λ or M configuration, and $\varepsilon = -1$ for V or W configuration.

The slowly varying probability amplitude c_j, $j = 1,\ldots,N$ of level j of the atoms and the slowly varying electromagnetic field amplitudes $\mathcal{E}_j$, $j = 1,\ldots,N-1$ of the $N-1$ lasers, are given in terms of the Schrödinger equation and Maxwell equations by

$$i\frac{\partial c_1}{\partial t} = -\tfrac{1}{2}\Omega_1 c_2, \tag{1a}$$

$$i\frac{\partial c_k}{\partial t} = -\tfrac{1}{2}\Omega_{k-1}c_{k-1} - \tfrac{1}{2}\Omega_k c_{k+1}, \quad k = 2,\ldots,N-1, \tag{1b}$$

$$i\frac{\partial c_N}{\partial t} = -\tfrac{1}{2}\Omega_{N-1}c_{N-1}, \tag{1c}$$

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial ct}\right)\Omega_k = 2i\varepsilon\mu_k c_k c_{k+1}, \quad k = 1,\ldots,N-1. \tag{2}$$

Here the Rabi frequencies are given by $\Omega_j = 2d_j\mathcal{E}_j/\hbar$, $j = 1,\ldots,N-1$, and the propagation coefficients are given by $\mu_j = 2\pi\mathcal{N}d_j^2\omega_j/\hbar c$, where d_j is the dipole matrix element between levels j and $j+1$, ω_j the corresponding laser frequency, c the speed of light, and $\mathcal{N}$ is the density of three- or five-level atoms in the medium. All dynamical variables are functions of both space z and time t. We have assumed one-photon resonances for all the allowed atomic transition and laser frequencies, and we have ignored decays to other levels.

If the pulses are shape invariant and propagate through the medium with velocity v, then they depend on t and z through $\xi = (t - z/v)/\tau$, where $\tau > 0$ is the pulse length. The initial value ξ_0 of ξ can be arbitrary. Two frequent choices of ξ_0 are 0 and $-\infty$. We assume that the propagation coefficients in (2) satisfy $\mu_1 = n\mu_2$, and for $N = 5$, we also assume $\mu_3 = \mu_2$, $\mu_4 = \mu_1$. The two independent pulses in our solutions are given in the form

$$\begin{aligned}\Omega_1(\xi) &= A_1 f_1(\xi) + A_2 f_2(\xi),\\ \Omega_2(\xi) &= B_1 f_1(\xi) + B_2 f_2(\xi),\end{aligned} \tag{3a}$$

where A_1, A_2, B_1, and B_2 are real constants, and f_1 and f_2 are dimensionless functions of ξ expressible in terms of Jacobi elliptic functions. For $N = 3$, $\Omega_1(\xi)$, and $\Omega_2(\xi)$ form the solitary-wave pair. For $N = 5$, we assume

$$\Omega_3(\xi) = \Omega_2(\xi), \quad \Omega_4(\xi) = \Omega_1(\xi), \tag{3b}$$

and thus there is in fact also only one pair of solitary waves. The atomic variables $c_j(\xi)$, $j = 1, \ldots, N$, are given by

$$c_1 = \frac{\tau}{R}(A_1 f_2 - \varepsilon A_2 f_1), \quad c_2 = \frac{2i}{R} f_3,$$
$$c_3 = \frac{n\tau}{R}(B_1 f_2 - \varepsilon B_2 f_1), \tag{4a}$$

and

$$c_4 = c_2 \text{ and } c_5 = c_1, \quad \text{for } N = 5, \tag{4b}$$

where R, a normalization constant, and the functions $f_1(\xi)$, $f_2(\xi)$, $f_3(\xi)$ will be given. The solitary-wave pair has a constant velocity v given by

$$1/v = 1/c + 4\mu_1 \tau^2/R^2. \tag{5}$$

The above specifications do not completely fix the analytic forms which the functions f_1, f_2, and f_3 can take. The solution which we shall present, which may be called the canonical solution, is one which other seemingly different analytic solutions can be reduced to by a simple transformation. First, for convenience, let us denote

$$a = \tfrac{1}{4}(A_1^2 + nB_1^2 + A_2^2 + nB_2^2),$$
$$b = \tfrac{1}{2}(A_1A_2 + nB_1B_2), \tag{6}$$
$$c = \tfrac{1}{4}(A_2^2 + nB_2^2 - A_1^2 - nB_1^2).$$

For our canonical solution, we assume $b = 0$ and $c \geq 0$. We give, in Table I, for the Λ or M configuration, two independent sets of solutions, marked (I) and (II), of the elliptic-function pairs f_1 and f_2, and for the V or W configuration, only one set of solution. Notice that f_1 and f_2 satisfy $f_2^2 + \varepsilon f_1^2 = 1$, and that $f_3 = \varepsilon f_1'/f_2$. The relationship between the squared modulus $0 \leq k^2 \leq 1$ (or the squared complementary modulus $k'^2 = 1 - k^2$) for these functions and the normalization constant R, and one other relationship which must be satisfied by the amplitudes and length of the pulses are given in the last two rows of Table I.

We first note that the combinations of pulse and field amplitudes given by $K_1 \equiv \Omega_2 c_1 - \Omega_1 c_3$ for $N = 3$ and $K_2 \equiv \Omega_2 c_1 - \Omega_1 c_3 + (\Omega_1\Omega_3/\Omega_4)c_5$ for $N = 5$ are constants of motion with $K_n = (n\tau/R)(A_1B_2 - A_2B_1)$ for any values of A's and B's. They are known to be constants of motion previously either (i) when the field amplitudes are time independent [3,4] or (ii) under the adiabatic condition for $N = 3$ when the pulses are incident in the counterintuitive order [5,6].

The solitary-wave pairs given by our general solution, Eqs. (3)–(5) and Table I, vary widely in shape and form: from ones for which the pulses have similar shape such as $\Omega_1 = A_2 f_2$, $\Omega_2 = B_2 f_2$ to ones which have very different but complementary shapes such as $\Omega_1 = A_2 f_2$, $\Omega_2 = B_1 f_1$. The pulse shapes include the special case $k = 0$ for which $\mathrm{sn}(\xi, 0) = \sin\xi$, $\mathrm{cn}(\xi, 0) = \cos\xi$, and $\mathrm{dn}(\xi, 0) = 1$ and the special case $k = 1$ for which $\mathrm{sn}(\xi, 1) = \tanh\xi$, $\mathrm{cn}(\xi, 1) = \mathrm{dn}(\xi, 1) = \mathrm{sech}\xi$. The solution offers a variety of initial conditions for the atoms. The parameters A_1, A_2, B_1, B_2, τ, and $0 \leq k^2 \leq 1$ are the free parameters at our disposal, subject to the relations $b = 0$, $c \geq 0$, and given in the last row of Table I. Once we made the above choice, the initial condition required for the atomic variables is determined by the values of $c_j(\xi_0)$, $j = 1, \ldots, N$, given by Eq. (4).

These solitary-wave pairs will be called collectively the matched-solitary-wave pairs (MSP). They exhibit a wide range of interesting properties only some of which we can describe in this short Letter. Consider first the MSP associated with a Λ or M configuration. It is useful to divide them into two groups even though the parameters characterizing them can be changed continuously from one to the other. The first group, called twin pairs (TP), is characterized by $|A_2| \gg |A_1|$ and $|B_2| \gg |B_1|$, which includes $A_1 = B_1 = 0$ as a special case. An example of TP for a three-level Λ system is shown in Fig. 1. The second group, called complementary pairs (CP), is characterized by either (i) $|A_2| \gg |A_1|$ and $|B_1| \gg |B_2|$, which includes $A_1 = B_2 = 0$ as a special case, or (ii) $|A_1| \gg |A_2|$ and $|B_2| \gg |B_1|$, which includes $A_2 = B_1 = 0$ as a special case. An example of CP for a three-level Λ system is shown in Fig. 2. From Table I, since $c\tau^2 = k^2$ or 1, thus for the TP, the value of $a\tau^2$ is restricted and $R^2 \approx 4n$. A consequence of this is that the area of the pulse pair over a pulse length cannot be greater than a certain value. Another consequence is that generally *all* levels of the atoms would at some time take up significant

TABLE I. Two independent sets, in columns marked (I) and (II), of elliptic-function pairs and other relevant quantities for Λ or M level configuration and one set of the corresponding quantities for V or W level configuration.

	Λ or M		V or W
	(I)	(II)	
f_1	$\mathrm{sn}(\xi, k)$	$k\,\mathrm{sn}(\xi, k)$	$(k/k')\mathrm{cn}(\xi, k)$
f_2	$\mathrm{cn}(\xi, k)$	$\mathrm{dn}(\xi, k)$	$(1/k')\mathrm{dn}(\xi, k)$
f_3	$\mathrm{dn}(\xi, k)$	$k\,\mathrm{cn}(\xi, k)$	$k\,\mathrm{sn}(\xi, k)$
R^2	$n[2(a\tau^2 - k^2) + 4]$	$n[2(a\tau^2 - 1) + 4k^2]$	$n[-2(c\tau^2 + k'^2) + 4]$
	$c\tau^2 = k^2$	$c\tau^2 = 1$	$a\tau^2 = k'^2$

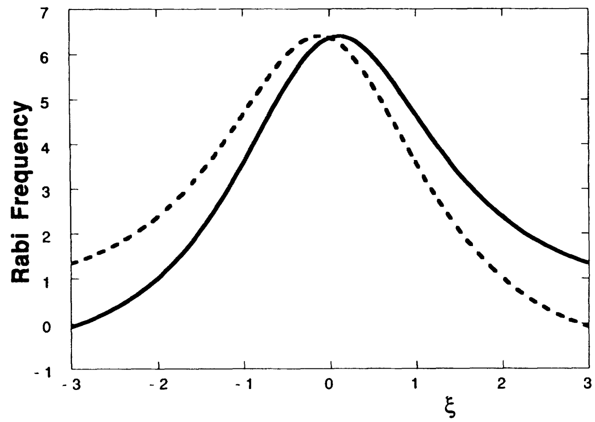

FIG. 1. Twin pair $\Omega_1 = (1/\sqrt{2})(\tanh\xi + 9\,\mathrm{sech}\xi)$ (solid line) and $\Omega_2 = (1/\sqrt{2})(-\tanh\xi + 9\,\mathrm{sech}\xi)$ (dotted line).

population during the pulse propagation. On the other hand, for the CP, the value of $a\tau^2$ and hence R^2 can be increased arbitrarily. A consequence of this is that the area of the pulse pair over a pulse length can be increased without limit. The two pulses "protect" each other, propagate with their shapes invariant, and do not break up into pulses of smaller areas. They are more general than EIT because both pulses can be strong laser pulses. Another important consequence is that during the propagation of a CP for which $a\tau^2 \gg 1$, the even-numbered levels would have insignificant occupation probability *at all time* during the pulse propagation. Relaxations from those levels, if present, can thus be reduced or eliminated. Numerical tests suggest furthermore that these CP become insensitive to small changes to their shapes.

The two classes of functions (I) and (II) in Table I give generally quite different atomic evolution for the CP propagation. We note that the sn and cn functions periodically become zero and change sign while the dn function is always positive and oscillates between its maximum value 1 and its minimum value $k' = (1 - k^2)^{1/2}$. Remembering that c_1 (and c_5 for $N = 5$) follows Ω_2 and c_3 follows Ω_1, CP of class I generally would completely deplete and then restore the population of level one (and five), while CP of class II generally would only partially deplete and then restore the population of level one (and five), as they propagate through the medium. However, the two classes coincide for the case $k = 1$. For this case, and for $N = 3$, the special case of $A_1 = B_1 = 0$ reduces the TP to two simultaneous pulses called simulton, obtained previously by Konopnicki and Eberly [7], and the special case of $A_1 = B_2 = 0$ reduces the CP to one given by Bol'shov and Napartovich [8], and Ackerhalt and Milonni [9].

Notice that for the CP propagation under the condition $a\tau^2 \gg 1$, c_k is small for even k, and the atomic evolution is well approximated for $N = 3$, by $c_1 \approx -\Omega_2/\mathcal{M}$, $c_2 \approx -2ic_1'/\tau\Omega_1$, and $c_3 \approx \Omega_1/\mathcal{M}$, where $\mathcal{M} = \{\Omega_1^2 + \Omega_2^2\}^{1/2}$, and for $N = 5$, by $c_1 \approx \Omega_2\Omega_4/\mathcal{M}$, $c_2 \approx -2ic_1'/\tau\Omega_1$, $c_3 \approx -\Omega_1\Omega_4/\mathcal{M}$, $c_4 = -2ic_5'/\tau\Omega_4$, $c_5 \approx \Omega_1\Omega_3/\mathcal{M}$, where $\mathcal{M} = \{\Omega_2^2\Omega_4^2 + \Omega_1^2\Omega_4^2 + \Omega_1^2\Omega_3^2\}^{1/2}$.

As a result of our understanding about CP, we found that many *approximate* CP solutions can be constructed under the condition $a\tau^2 \gg 1$. The pulse pair is given generally by $\Omega_1 = Af$, $\Omega_2 = B(1 - f^2)^{1/2}$, $|f(\xi)|_{\max} < 1$, $A\tau$ and $B\tau \gg 1$, and $A = \sqrt{n}B$. The speed v of propagation is given by $1/v - 1/c \approx 4\mu_1/nA^2$, and the atomic evolution is given by $c_1 \approx -(1/\sqrt{n})(1 - f^2)^{1/2}$, $c_2 \approx -(2i/\sqrt{n}A\tau)f'/(1 - f^2)^{1/2}$, $c_3 \approx f$, and $c_4 = c_2$, $c_5 = c_1$ for $N = 5$. An example of this approximate CP is a Gaussian-complementary-Gaussian pair, given by $\Omega_1 = A\alpha\exp(-\xi^2)$, $\Omega_2 = B[1 - \alpha^2\exp(-2\xi^2)]^{1/2}$, where $0 < |\alpha| < 1$. These approximate CP have the same form as our analytic CP, but the functions f_1 and f_2 are not ones given in Table I from our analytic solution.

For the V or W configuration, the constraints on $a\tau^2$ and $c\tau^2$ from Table I are such that we have an analogous group of TP but not CP. For the case $k \to 1$, the limiting TP is obtained by making the A's and B's approach zero in such a way that $\lim(A_1 + A_2)/k' = A$, $\lim(B_1 + B_2)/k' = B$, so that $\Omega_1 = A\,\mathrm{sech}\xi$, $\Omega_2 = B\,\mathrm{sech}\xi$, with $(A^2 + nB^2)\tau^2 = 4$, and $R^2 = 4n$. For $N = 3$, this particular TP is another simulation found previously by Konopnicki and Eberly [7].

It is curious to note that for the case $k = 1$, while our MSP solution allows solitary-wave pairs given quite generally by $\Omega_1 = A_1\tanh\xi + A_2\,\mathrm{sech}\xi$ and $\Omega_2 = B_1\tanh\xi + B_2\,\mathrm{sech}\xi$ for the Λ and M configuration, it allows only solitary-wave pairs of a more restricted form: $\Omega_1 = A\,\mathrm{sech}\xi$ and $\Omega_2 = B\,\mathrm{sech}\xi$ for the V and W configuration. Shape-preserving solitary waves which have an analytic form other than a simple hyperbolic-secant shape is rare. Another known example was given by one of us [10] who obtained an analytic form for a group of four simultons that can propagate through an $N = 5$ system with a M or W configuration.

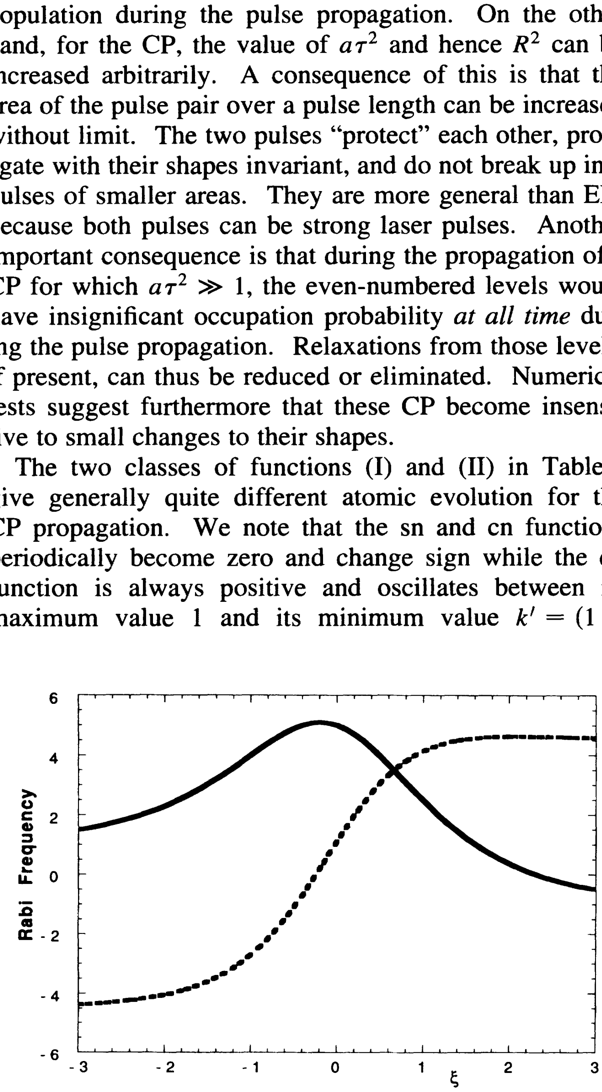

FIG. 2. Complementary pair $\Omega_1 = -\tanh\xi + 5\,\mathrm{sech}\xi$ (solid line) and $\Omega_2 = 4.5\tanh\xi + 1.11\,\mathrm{sech}\xi$ (dotted line).

Their Rabi frequencies are $\Omega_j = \sqrt{j(N-j)}\Omega_0$ for $j = 1, \ldots, 4$, where $\Omega_0 = (\sqrt{8}/\tau)\,\mathrm{sech}\xi/(1 + 7\,\mathrm{sech}^2\xi)$.

In summary, we have presented exact analytic results that give a great variety of solitary-wave pairs that can propagate through a three-level system and a five-level system without changing their shapes. The MSP range from CP, on the one hand, to TP, on the other hand, and all the variation in between. The required initial atomic condition varies widely, and the resulting atomic evolution ranges from complete depletion and return of population to very insignificant changes in the occupation probabilities of some levels. Even in cases when the population of some levels evolve significantly, the occupation probability of some intermediate levels of the atoms can be made small, even though all transitions are at one-photon resonance. We find it remarkable that our MSP have such variety of forms and shapes with such different degree of robustness not found before in the studies of solitary pulse propagation.

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