The Review: A Journal of Undergraduate Student Research

Volume 23 Article 7

2022

How to Guard an Art Gallery: A Simple Mathematical Problem

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Abstract

The art gallery problem is a geometry question that seeks to find the minimum number of guards necessary to guard an art gallery based on the qualities of the museum's shape, specifically the number of walls. Solved by Václav Chvátal in 1975, the resulting Art Gallery Theorem dictates that $\lfloor n/3 \rfloor$ guards are always sufficient and sometimes necessary to guard an art gallery with n walls. This theorem, along with the argument that proves it, are accessible and interesting results even to one with little to no mathematical knowledge, introducing readers to common concepts in both geometry and graph theory. Furthermore, the Art Gallery Theorem and its proof have many extensions, leading to other related theorems on guarding galleries, as well as various applications, including the use of its methods in robotics and GPS. This paper serves as a cursory introduction to the theorem, its most commonly referenced proof, and these extensions and applications, particularly for those with little familiarity with visibility problems or mathematics in general.

Keywords

art gallery theorem, art gallery problem, computational geometry, visibility problem, geometry, graph theory, math, mathematics, math education, education, logic, proofs

Cover Page Footnote

Thank you to Dr. Erica Johnson, Dr. Mark McKinzie, Dr. Ryan Gantner, and Dr. Kris Green for their assistance in directing the focus of this research. Thank you to my assiduous editors, Katelyn E. Coykendall, Timothy Stoklosa, and Marianne Petruzelli. Finally, thank you to Rosemary Howard, for whom this paper is written.

How to Guard an Art Gallery: A Simple Mathematical Problem

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Abstract: *The art gallery problem is a geometry question that seeks to find the minimum number of guards necessary to guard an art gallery based on the qualities of the museum's shape, specifically the number of walls. Solved by Václav Chvátal in 1975, the resulting Art Gallery Theorem dictates that* $\lfloor n/3 \rfloor$ *guards are always sufficient and sometimes necessary to guard an art gallery with n walls. This theorem, along with the argument that proves it, are accessible and interesting results even to one with little to no mathematical knowledge, introducing readers to common concepts in both geometry and graph theory. Furthermore, the Art Gallery Theorem and its proof have many extensions, leading to other related theorems on guarding galleries, as well as various applications, including the use of its methods in robotics and GPS. This paper serves as a cursory introduction to the theorem, its most commonly referenced proof, and these extensions and applications, particularly for those with little familiarity with visibility problems or mathematics in general.*

Imagine being in an art gallery. It can be any shape, the more complicated the better. How many guards are there? Where are they positioned? Is every part of the gallery guarded at all times, so that each part of the museum is in the line of sight of a guard? Or is there an area that no guard is able to see? This is the start of a well-known mathematical quandary called the art gallery problem, which seeks to find the minimum number of guards necessary to protect a museum of simple polygonal shape. Despite the perceived complexity of this problem, its solution is surprisingly simple, providing both a number of guards necessary for any particular gallery as well as describing a process with which one can find where the guards should be positioned. The resulting deduction and its process are known as the Art Gallery Theorem, and its notability is well-deserved in more fields than just mathematics, requiring little familiarity with math to understand while still introducing and utilizing a broad variety of mathematical methods. Furthermore, in the decades since the original art gallery problem was posed, the relatively simple result has been expanded in a multitude of different directions, inspiring many variations of the theorem to model real-life situations.

History and Mathematical Background

This problem was first proposed in 1973 by mathematician Victor Klee to Václav Chvátal, who had asked Klee to surprise him with an interesting geometry problem to solve (O'Rourke 1). Chvátal produced a solution to this question two years later, which became knownas Chvátal's Art Gallery Theorem. The *Art Gallery Theorem* states that $\lfloor n/3 \rfloor$ guards are always sufficient and sometimes necessary to guard a simple polygonal art gallery with n walls (O'Rourke 9). Given the sheer number of shapes an architect could make an art gallery, this theorem likely seems deceptively simple. Indeed, there are many components to this statement that ultimately combine to make a very straightforward solution, but they are not difficult to grasp as individual building blocks.

The art gallery problem is known as a visibility problem, as it concerns what the gallery guards are able to see. A guard is able to guard, or "cover," any part of the gallery that is within their field of vision. Any guard may rotate 360° and thus protect any space that they can see around them. Moreover, the shape of the art gallery must be a simple polygon, like a triangle or an octagon, which is a figure constructed out of lines that do not intersect itself. In this case, the polygon in question is given by the floor plan of the art gallery. The number of sides, or walls, of the shape is *n*, a variable. This ambiguity is given so that one can analyze all possible shapes with *n* sides; for instance, if $n = 5$, the theorem should give us the number of guards necessary for all possible pentagon shaped galleries. The minimum number of guards needed to guard all polygonal art galleries is thus given by the formula $\lfloor n/3 \rfloor$. The formula is inside of what is called a "floor function." The floor function essentially truncates decimals, which means that if the number produced by $n/3$ is a decimal, then it should always be rounded down. Using the same example as before, if the gallery has 5 walls, then 5 divided by 3 is clearly a decimal, equal to approximately 1.667. The floor function therefore rounds that value down to 1 guard for a pentagon shaped gallery.

In fact, the most complicated part of this theorem is not the expression, but the language. The phrase "always sufficient and sometimes necessary" may seem verbose, but the added caveat gives this simple theorem a completely new meaning. For instance, picture an art gallery with 27 walls. There are a great deal of different ways to imagine what a 27-walled room would look like. Perhaps the walls are arranged so that the room is shaped roughly like a circle, in which case only one guard right in the middle would be needed. However, the walls might also be arranged in a maze-like fashion, filled with narrow and twisting corridors such that one guardwould never be able to see every part of it at once. Chvátal's theorem accounts for both of these cases and all the possibilities between them by providing an "upper bound," which essentially means the largest value the minimum number of guards will ever be.

Thus, the value $\lfloor n/3 \rfloor$ will be the number of guards required — or "sometimes necessary" — for the proverbial worst-case art gallery with *n* walls. If the gallery's shape is far simpler, then that value will be "always sufficient" and possibly fewer guards may suffice. The Art Gallery Theorem thus elegantly combines all these pieces into one simple result for all galleries with a given number of walls.

Proving the Theorem: An Outline

With all this ambiguity, the theorem likely seems quite difficult to prove. In mathematics, a result is considered proven when an argument (known as a proof) is given that shows the statement is true for all possible situations; in this case, a proof would have to show that the theorem is true for all possible art galleries with any number of walls. This may seem excessive, but luckily enough, it is sufficient to show a valid process — applicable to any gallery shape — which gives the number of guards that can guard the entire gallery and is never greater than $\lfloor n/3 \rfloor$. There was initially no easily accessible proof to this theorem. Chvátal's original proof was extremely complex, using a method called "induction" in order to rigorously prove the result (O'Rourke 6). His method has been popularly described as unconventional and not intuitive, and is not easily understood even to one with a mathematics background. Chvátal's work was later eclipsed by a mathematician named Steve Fisk, whose proof of the theorem was so ingenious and simple that it was included in "Proofs from THE BOOK," a book where only the most elegant of mathematical results are recorded (Chesnokov). Fisk's proof was so inventive that his process is accessible even to those with little knowledge of geometry, particularly given its visual approach to each gallery, which even shows potential locations where guards should be placed.

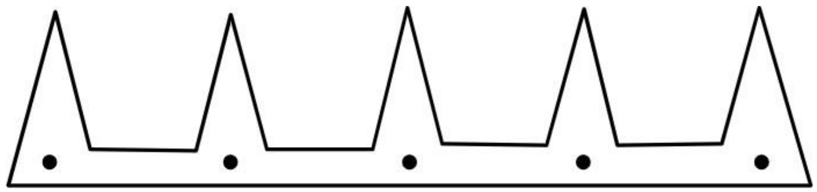

Figure 1. A comb with 5 prongs, each requiring a guard (shown as dots).

First, an important shape to note is the "comb," shown in figure 1, which has protrusions called "prongs." The comb was found by Chvátal to be the proverbial worst-case scenario for a gallery with n walls because for any prong of the comb, a guard would be required at the base ofit, as no guard can have two prongs in his line of sight at once (Michael and Pinciu). Thus, the minimum number of guards needed would be the same as the number of prongs. Adding anotherprong to a comb (and thus another guard) would require no fewer than 3 additional walls, and so the minimum number of necessary guards for this comb shape would be $\lfloor n/3 \rfloor$ (Michael and Pinciu). The expression $n/3$ is the approximate number of prongs in the comb, presuming that they need three walls to be formed, while the floor function is necessary in case the number of walls is not a multiple of 3. Therefore, the number of guards needed to guard the most complexpossible gallery of n walls is no less than $\lfloor n/3 \rfloor$. Fisk's process then aims to show that no more than $\lfloor n/3 \rfloor$ will be ever required for any art gallery, as well as where to place them.

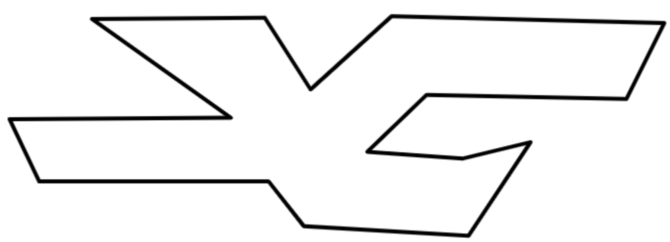

Figure 2. An art gallery floor plan with 16 walls.

The process for finding the necessary number of guards for a gallery and their position, as given by Fisk, can be explained in three relatively simple steps. It will be illustrated on the X-shaped gallery pictured in figure 2, which has 16 walls. Chvátal's theorem says that a gallery of this arrangement should need at most 5 guards ($\lfloor 16/3 \rfloor = 5$, where $n = 5$), and Fisk's process should confirm this. The first step is called triangulation, and as implied, it includes dividing upthe gallery so that it is in triangles, producing what is called a triangulation graph (O'Rourke 4).This is a bit more complicated than described; the corners of the gallery's shape can also be called vertices, and diagonal lines must be inserted between the vertices so that triangles are produced without the diagonals intersecting one another (Michael and Pinciu). There are potentially many unique triangulations of a shape, but any one will suffice for this process (Michael and Pinciu). The latter is pictured on the example gallery in figure 3.

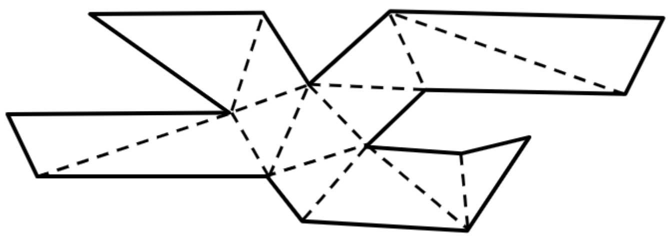

Figure 3. The triangulation graph of figure 2, with 13 diagonals.

The second step is to find what is called a 3-coloring of the triangulation graph. A 3-coloring is a way of assigning 3 colors to the vertices of a graph so that no two vertices that are connected by a wall or one of the inserted diagonals are colored the same color (O'Rourke 5). In simpler terms, if the corners of the art gallery have a wall or a diagonal line between them, they cannot be colored the same color. Again, there may be multiple valid 3-colorings to the gallery, but one is shown below in figure 4. It is relevant to note that this process assumes that all simple polygons have a triangulation graph and that these graphs have a 3-coloring. These assumptions are correct, but as aforementioned, would require their own proofs in order to show both are true for all polygons. To maintain relative simplicity, these assumptions will be considered true as if already proven, but the reader is encouraged to explore why this might be so.

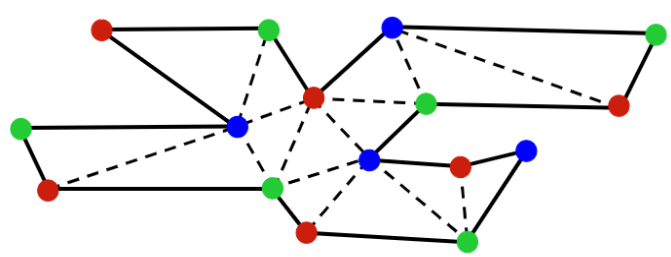

Figure 4. The 3-coloring of figure 3.

The final step is to count the number of vertices of each color and select the least frequently used color (O'Rourke 6). Since there are three colors, it is not possible for more than two colors to each be used over ⅓ of the time — else there would be greater than *n* vertices colored — and so one color must be used no more than ⅓ of the time (O'Rourke 6). Therefore, the least frequently used color would not be used more than $\lfloor n/3 \rfloor$ times. If two or more of the colors appear an equal number of times, one may be selected at random. The guards can be placed at the vertices that are colored the least used color, and the process is complete (O'Rourke 6). Furthermore, the proof has illustrated that $\lfloor n/3 \rfloor$ guards is both necessary, in the case of the worst-case comb, and sufficient, as no gallery of *n* sides will require more than this number of guards, to guard a museum of *n* walls.

Inherent in this final step is the assumption that a guard is able to protect, or cover, all of the triangles they are adjacent to, and if there is at least one guard adjacent to every triangle in the graph, then the entire gallery floor is protected (O'Rourke 6). In the example gallery, the least frequently used color is blue, which appears four times, and so the guards can be placed there. As previously stated, the theorem projected that this gallery would need at most 5 guards, and the process confirmed this, showing that the gallery only needed 4 guards, which is obviously within the anticipated range of results.

This essentially describes the steps necessary to determine both the upper bound of the number of guards needed for a gallery and their positions. The process is quite manageable, even though it employs techniques from geometry and graph theory, like triangulation and vertex coloring. Compared to the complexity of Chvátal's work, any individual with a basic knowledge of geometry would be able to complete the triangulation step, and the vertex coloring step would only require a slight understanding of graph theory — or an interest in simple puzzles. Thus, the Art Gallery Theorem is an incredibly accessible result and introduces the reader to a wide range of mathematical material with little prior knowledge required.

Extensions and Applications

Chvátal and Fisk's proofs of the Art Gallery Theorem served as the seminal work for a variety of other results. Though the focus of the theorem may seem quite narrow, there are a great deal of expansions known collectively as the "art gallery theorems." These include variations in which the shape of the gallery is restricted or generalized further, or the abilities and tasks of the guards are changed (Michael and Pinciu). Some of these theorems feature polygons with holes (displays or columns in the art gallery that block the lines of sight of the guards) or allow guards to move throughout the gallery on rounds (Michael and Pinciu). Others delve deeper into graph theory and seek to find bounds for a chromatic number for all art galleries of *n* walls, which is the smallest number of colors ever needed to color a gallery according to the above specifications (Chesnokov). Additionally, some might restrict a guard's field of vision to only 180°, so they must have their back to the wall at all times, or demand the guards be present on only walls of the gallery and not corners (Michael and Pinciu). All of these variations present their own puzzle, and each have different and interesting proofs, though none quite as simple as the original Art Gallery Theorem.

Despite the obvious and varied applications of this to the security of art museums, the Art Gallery Theorem does hold some relevance in other fields of study, though it is usually the method of proof and not the result itself. The Art Gallery Theorem was solved via computational geometry, including the usage of discrete math and optimization, in order to determine the most efficient use of a plane — in this case, the floor plan of an art gallery (Moise). This methodology can be used in many situations; for instance, GPS route planning, where computational geometry plays an integral role in deciding speed, direction, location, and routes for a vehicle (Moise).

Additionally, similar techniques of calculation are heavily utilized in robotics, where machine algorithms must be able to sense and avoid obstacles, as well as determine which routes are open or blocked (Moise). Finally, another common application of computational geometry is in video games, where features such as object collision detection, realistic player motion, and how objects appear in virtual reality all rely on these mathematical constructs (Moise).

Furthermore, the proof techniques used in the Art Gallery Theorem can be utilized elsewhere, in so-called “art gallery problems” (Michael 89). One example of this is called the zookeeper problem, where the aim is to find the shortest possible path inside a zoo that meets the boundary of each animal’s cage, with the cages and the zoo being represented by smaller polygons contained within a larger one. (Michael 89). This route illustrates a safe and efficient path for the zookeeper during feeding time, and Chvátal’s inductive approach can be applied to solve this type of problem. The zookeeper problem is relevant in real-life situations as well; scientists directing a rover on Mars experience a type of zookeeper problem when they move the rover to collect data, except they have constraints on time, energy, and terrain (Michael 89). There are several other problems like this, including some called the fortress and prison yard problems, which employ the same methods and can be seen as extensions of the art gallery problem (Michael 106).

Concluding Remarks

The art gallery problem is an excellent introduction to the world of mathematical arguments, transforming what might seem like an inane question with little need for numbers to a unique puzzle that gives a synopsis of various mathematical fields and has several unexpected real-world applications. The Art Gallery Theorem exemplifies how even simple results can be expanded into a subfield of research outside of mere security arrangements, with new theorems and problems that each singularly interpret the world in a slightly different manner, providing unique perspectives to problems like the Mars rover. Moreover, the art gallery problem is significant because of its simplicity and succinctness, demonstrating a friendly and visual introduction to an interesting math problem. With relatively little effort, even someone with no experience in mathematics could comprehend the process involved in solving this problem and glean a new understanding of mathematical reasoning, proof, geometry, and perhaps most importantly, the best way to guard their art galleries.

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