The Impact of van Hiele-based Geometry Instruction on Student Understanding

Susan Connolly
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The Impact of van Hiele-based Geometry Instruction on Student Understanding

Abstract
Developments over the last three decades provide momentum for revising high school geometry instruction as recommended by the van Hieles. Cognitive learning theories, brain research, multiple intelligence theories, revised national and state standards and computer technology-based tools all contribute to the rationale and the means to deliver instruction that enables students to construct knowledge and understanding through a sequential process of exploration, inductive and deductive reasoning. A Regents Geometry unit on quadrilaterals was developed based on these theories and techniques. Forty-three students enrolled in the high school Regents Geometry course received instruction using the newly developed materials. The results of these students showed improvement over the results of the previous year's students under more traditional geometry instruction.

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The Impact of van Hiele-based Geometry Instruction on Student Understanding

By

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Supervised by

Dr. Diane Barrett and Dr. Bernard Ricca

School of Arts and Sciences

St. John Fisher College

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Abstract

Developments over the last three decades provide momentum for revising high school geometry instruction as recommended by the van Hieles. Cognitive learning theories, brain research, multiple intelligence theories, revised national and state standards and computer technology-based tools all contribute to the rationale and the means to deliver instruction that enables students to construct knowledge and understanding through a sequential process of exploration, inductive and deductive reasoning. A Regents Geometry unit on quadrilaterals was developed based on these theories and techniques. Forty-three students enrolled in the high school Regents Geometry course received instruction using the newly developed materials. The results of these students showed improvement over the results of the previous year’s students under more traditional geometry instruction.
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Traditionally, high school geometry has been taught from an axiomatic framework, much the way that Euclid’s *Elements* documented geometric learning in 300 BCE. Many students struggle with geometry and do not perceive its value. Teachers become frustrated with the slow progress of most students. Yet the word *geometry* translates to the measure of space, an extremely relevant subject full of real world connections.

The purpose of this study is to understand the scope of the knowledge and skills that learners must develop to master geometry, explanations for the frustrations with geometry, and the pedagogical and instructional elements that optimize student learning.
Literature Review

Geometry is one of the earliest forms of mathematics and a well-established component of current mathematics education. The first geometry textbook is commonly accepted to be Euclid’s *Elements*, compiled around 300 BCE. Formal geometry instruction has historically followed Euclid’s axiomatic system and deductive reasoning approach. Coincidentally, or as many believe, consequently, geometry is one of the least understood and most disliked mathematics subjects. It seems contradictory that a subject with such direct real-world connections and immediate relevance can be such a source of frustration for students and teachers. The purpose of this study is to understand the scope of the knowledge and skills that learners must develop to master geometry, explanations for the frustrations with geometry, and the pedagogical and instructional elements that optimize student learning.

First, the van Hiele model of learning geometry will be discussed followed by the results of subsequent United States academic research into the van Hiele theories. Other perspectives on the learning of geometry will then be presented. These perspectives primarily address the breadth of skills involved in geometry. From a more comprehensive mathematics instructional viewpoint, the impact of affective learning issues must be considered, along with the processes of creating and accessing knowledge. Finally, the role of computer technology in learning geometry is a relatively recent development that has tremendous potential and connection to geometry instruction. With this foundation, the objectives and expectations for student achievement in the recently defined New York Regents Geometry course are then analyzed summarized in order to compile a current picture of the growth required of students during the Geometry course.
Van Hieles’ Theories on the Stages of Geometric Thinking

Reflecting upon years of teaching geometry in the Netherlands in the 1950’s, Dina and Pierre van Hiele researched the nature of learning geometry and the reasons for students’ struggles with the subject. Their research resulted in a model of learning stages and a recommendation for the sequence of instructional experiences that would enable a student to progress through the levels.

The van Hiele learning model consists of five levels. These levels are concisely described by Mason (1997):

*Level 1 (Visualization):* Students recognize figures by appearance alone, often by comparing them to a known prototype. The properties of a figure are not perceived. At this level, students make decisions based on perception, not reasoning.

*Level 2 (Analysis):* Students see figures as collections of properties. They can recognize and name properties of geometric figures, but they do not see relationships between these properties. When describing an object, a student operating at this level might list all the properties the student knows, but not discern which properties are necessary and which are sufficient to describe the object.

*Level 3 (Abstraction):* Students perceive relationships between properties and between figures. At this level, students can create meaningful definitions and give informal arguments to justify their reasoning. Logical implications and class inclusions, such as squares being a type of
rectangle, are understood. The role and significance of formal deduction, however, is not understood.

Level 4 (Deduction): Students can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions. At this level, students should be able to construct proofs such as those typically found in a high school geometry class.

Level 5 (Rigor): Students at this level understand the formal aspects of deduction, such as establishing and comparing mathematical systems. Students at this level can understand the use of indirect proof and proof by contrapositive, and can understand non-Euclidean systems. (pp. 4-5)

A student must progress through these levels sequentially. At each level, what is intrinsic at one level becomes explicit at the next level. Each level has its own set of terminology, symbols, concepts and reasoning strategies. It naturally follows that two people at different levels have difficulty understanding each other. It is difficult for someone at a higher level to recall what it’s like to think at a lower level, while the person at the lower level has not yet developed the foundation to make sense of the language and thought processes of the higher level (Usiskin & Senk, 1982).

The van Hieles proposed that students progress sequentially from one level to the next by working through instructional activities that are appropriate in terms of language and task for their level of understanding. Fuys, Geddes and Tischler (1988) describe the van Hieles’ five instructional phases as:

Information: The student gets acquainted with the working domain (e.g., examines examples and non-examples).
**Guided orientation:** The student does tasks involving different relations of the network that is to be formed (e.g., folding, measuring, looking for symmetry).

**Explicitation:** The student becomes conscious of the relations, tries to express them in words, and learns technical language which accompanies the subject matter (e.g., expresses ideas about properties of figures).

**Free orientation:** The student learns, by doing more complex tasks, to find his/her own way in the network of relations (e.g., knowing properties of one kind of shape, investigates these properties for a new shape, such as kites).

**Integration:** The student summarizes all that he/she has learned about the subject, then reflects on his/her actions and obtains an overview of the newly formed network of relations now available (e.g., properties of a figure are summarized). (p.7)

Van Hiele’s theories build upon but diverge from Piaget’s theories of learning. Both describe stages of thinking progressing from the visual and concrete to the more abstract and analytical. Whereas Piaget believed that children’s cognitive abilities are more dependent on biological maturation, van Hiele believed that development of the abstract and higher levels of thinking that are inherent in geometry are strongly influenced by the type and quality of instructional experiences (van Hiele, 1999).

**United States Reactions to van Hiele Theories**

Whereas the Soviet Union restructured geometry education based on the van Hiele theories in the 1960’s (Fuys, Geddes & Tischler, 1988), it was not until the 1980’s
that the United States began to take interest. In 1980, the National Science Foundation funded the translation of the van Hiele publications and three major projects to investigate the van Hiele model (Fuys et al., 1988).

Usiskin and Senk (1982) were the leaders of the Cognitive Development and Achievement in Secondary School Geometry (CDASSG) Project. Their objective was to study the relationship between van Hiele level and achievement in students enrolled in a one-year high school geometry course. The challenge was to take an elegant theory and translate it into a practical and predictive assessment tool. In order to do so, the team developed three testing instruments to gauge student performance levels in incoming geometry knowledge, van Hiele level and geometric proof.

The CDASSG team concluded that the assessment tools they had created were predictive of van Hiele level for two-thirds to nine-tenths of the students who participated. They also found that the van Hiele level was predictive of results on the content-based multiple-choice test. They found that while van Hiele level correlated to proof test results, the better predictor of proof test performance was the score on the content-based test. Another interesting finding was that among students starting at the same van Hiele level in the fall assessment, there was great variability in the extent of change in van Hiele level at the end of the course. According to Usiskin and Senk (1982), “as one would expect, there are factors other than van Hiele level operating to contribute to growth in understanding in geometry” (p. 81). Arguably the most startling conclusion is the following: “In geometry classes that study proof, the fall van Hiele levels of over half the students are too low to afford even a 2 in 5 chance of success at
proof.” (p. 84). Those students at a low level 2 or less had low probability of understanding proof.

Burger and Shaughnessy (1986) developed an assessment tool that involves a structured interview focused on triangles and quadrilaterals. Their interviews of 45 students in Oregon, Michigan and Ohio confirmed that the van Hiele levels are useful in describing students’ thinking processes regarding geometry. They concluded that their structured interview process provided insights and consistent conclusions regarding a student’s van Hiele level of understanding.

The Van Hiele Model of Thinking in Geometry among Adolescents project led by Fuys, Geddes and Tischler covered a great deal of ground, generally more focused on the instructional aspects of the van Hiele theory (Fuys et al., 1988).

The general question that this research addressed is whether the van Hiele model describes how students learn geometry. There were four main objectives:

(1) To develop and document a working model of the van Hiele levels, based on several sources which the Project had translated from Dutch into English.

(2) To characterize the thinking in geometry of sixth and ninth graders in terms of levels— in particular, at what levels are students?, do they show potential for progress within a level or to a higher level?, and what difficulties do they encounter?
(3) To determine if teachers of grades 6 and 9 can be trained to identify van Hiele levels of geometry thinking of students and of geometry curriculum materials.

(4) To analyze current geometry curriculum as evidenced by American text series (grades K-8) in light of the van Hiele model. (p.1)

The second objective involved studying two groups of very academically and racially diverse students from New York City public schools. The instructional interventions were not part of normal classroom activities for the sixteen sixth and sixteen ninth graders and the topics chosen were ones that would be new to the students. The instructor-interviewers worked with students over a three-week time frame. Whereas CDASSG employed a very quantitative approach with a large sample, this project assessed levels through more in-depth, qualitative, interactive interviews.

The report discussed a variety of factors that affected student performance on the modules. Issues pertaining to prior math learning experiences included: the student’s degree of familiarity with the appropriate geometry vocabulary; misconceptions introduced from incomplete or flawed prior learning experiences; and perceptual difficulties in recognizing elements in diagrams, sometimes affected by diagram orientation. Student learning styles or affective factors included: student’s views about how to learn mathematics, comfort with exploration versus waiting for a fact to memorize; student’s degree of confidence, persistence, independence as a learner; students’ tendencies for impulsive versus reflective conclusions (Fuys et al., 1988).

It was observed that:
All subjects made extensive use of the concrete materials provided to explore relationships, discover patterns, or confirm hypotheses. The use of manipulatives and other concrete materials allowed the students to try out their ideas, look at them, be reflective, and modify them. The visual approach seemed not only to maintain student interest but also to assist students in creating definitions and new conjectures, in gaining insight into new relationships and interrelationships. (p. 138)

The results of these three projects introduced the United States to the van Hieles’ conceptual model and triggered a wave of investigation, reaction and response. (Baynes, (1998); UCSMP, (n.d.); Yazdani, (2007)).

**Other Perspectives on Learning Geometry**

Alan Hoffer (1981) captured an important perspective in the title of his article: *Geometry is more than Proof*. While proof is an important component, there are several other valuable skills that should be nurtured within the geometry curriculum. Hoffer categorized the content of geometry into five skills: visual, drawing, verbal, logical and applied. He integrated these skills as a second dimension to the van Hiele levels. He proposed that instruction should support student advancement through van Hiele levels in each of the dimensions of geometric skills. Each of these skill areas present their own instructional challenges but have life-long value beyond the geometry classroom.

Along similar lines, the AIMS Education Foundation’s Model of Learning Math and Science (figure 1) emphasized the four environments of learning about the world: real world manipulatives, aligned with visual skills; representation through graphs, illustrations or diagrams, aligned with drawing skills; abstraction through reading and
writing numbers and words, aligned with verbal skills; and further abstraction through thinking, analyzing, generalizing, hypothesizing and applying, aligned with logic and applied skills. The arrows between environments reinforce the importance of developing students’ ability to connect and transfer between environments. It is the Foundation’s opinion that too much time is spent working with words and numbers, at the expense of developing the other thinking dimensions (AIMS Education Foundation, 2006).

Visual skills relate to spatial skills and visual perception. Howard Gardner defined spatial intelligence as “capacities to perceive the visual-spatial world accurately and to perform transformations on one’s initial perceptions” (1989, p. 6). There appear to be two components to visual skills that impact geometry learning. One component is the ability to discern relevant information from a physical object or diagram. The other is to be able to visualize changes to the given object or diagram.

Fuys, Geddes and Tischler (1986) observed that visual perception was a key factor affecting geometry learning in the students they studied. They stated that orientation and figure-ground issues affected what the students observed. Burger and Shaughnessy (1986) observed that turning or moving objects into more standard positions helped students identify right angles, parallel lines and congruent figures. Tartre (1990) used the term, spatial orientation to describe this task of visual perception:

…those tasks that require that the subject mentally readjust her or his perspective to become consistent with a representation of an object presented visually. Spatial orientation tasks could involve organizing, recognizing, making sense out of a visual representation, reseeing it or
Being able to visualize changes to a given object or diagram is the skill of spatial visualization. According to Tartre, “Spatial visualization is distinguished from spatial orientation tasks by identifying what is to be moved; if the task suggests that all or part of a representation be mentally moved or altered, it is considered a spatial visualization task” (p. 219). Geometry topics such as transformations and three-dimensional solids rely heavily on spatial visualization skills.

Battista (1994) proposed that visual-spatial processing is the initial stage in developing conceptual understanding of mathematical ideas, even as fundamental as addition. The learning sequence begins with concrete manipulation, moves into the ability to mentally manipulate images, progresses to the development of language to describe the operation and the result, and with repeated experience, leads to symbols representing the operations, “bypassing the spatial-like thinking required to use the underlying mental model. However, even though such thinking may appear strictly verbal, for it to be conceptually meaningful and powerful enough to encompass novel situations, it must be based – at some level – on operations with mental models” (p. 93). Therefore, it is important to explicitly include visual representations as students learn new content in order to build a foundation for conceptual understanding and a basis for problem solving.

Drawing skills are essential for the ability to capture the key elements of a problem represented in the physical world or represented through a verbal description. According to Hoffer (1981), “drawing skills can and probably should be developed in
geometry courses, and the activities often help prepare students to learn geometric relationships later in the course” (p. 12). Drawings can be more efficient than language for communicating the relationships of geometric objects. It has been the author’s experience that students’ incoming abilities to draw geometric configurations vary greatly based on prior extracurricular and academic experiences and often require explicit instruction.

Hoffer’s category of verbal skills has several dimensions. Geometry involves an extensive list of vocabulary. Many of these terms are new or used differently than in other contexts. Even the concept of a definition is different in geometry: a good definition must classify a term as well as differentiate it from other similar terms. Reading comprehension is essential to interpret theorems and proofs. Writing ability is necessary to articulate observed patterns and create explanations used in informal and formal proof. Reading and writing in a geometry context need more explicit instruction than in other math courses.

Logical thinking is a complex task. Tartre (1990) summarized brain researchers’ conclusions on the types of logical thinking processes.

…there are at least two types of logical thinking processes: one type that is characterized by step-by-step, deductive, and often verbal processes and one type that suggests more structural, global, relational, intuitive, spatial, and inductive processes. (p. 219)

Clearly, verbal skills overlap into logical skills. Many geometry problem-solving tasks involve accessing definitions and theorems to justify conclusions. Developing or analyzing a logical argument requires precise use of vocabulary and language. Perhaps
more surprisingly, spatial thinking skills connect to logical reasoning as well. Tartre’s (1990) research also showed that it is the spatial orientation skill, the ability to focus on certain details in an image, that most directly relates to logical reasoning. As the van Hiele levels described, the development of deductive reasoning in geometry represents a higher level of ability, dependent upon several prior stages of development. Focus on the verbal and visual/spatial skills should develop the base for logical reasoning.

Hoffer interpreted applied skills to mean the relation of mathematical models to physical phenomena (1981). As modeling skills are developed, students can be exposed to the practical applications of geometry. This adds relevance and enhances student appreciation of the geometry content.

Usiskin and Senk (1982) commented, “there are factors other than van Hiele level operating to contribute to growth in understanding in geometry” (p. 81). These factors might be the types of skills described by Hoffer, they might be related to a student’s affective learning characteristics, or they might be related to a student’s ability to access previously learned information. Lawson and Chinnappan (2000) contended that:

…failure to access available knowledge might arise from three aspects of students’ processing activity: the students’ dispositional states, the strategic nature of their memory-search activity, and the quality of organization of the knowledge relevant to the problems being considered. Thus, by way of illustration, access failure might result from one or more of the following problems: lack of persistence with the solution attempt due to low self-efficacy, ineffective use of cues provided in the problem
statement, or lack of strongly connected knowledge relevant to the problem (p. 28).

Interpretation of a problem relates to the level of visual and verbal skills of a student and was discussed above. The roles of affective issues and effective access to previous learning will be discussed below.

Affective Issues in Learning Geometry

A great deal of research has focused on the cognitive side of learning. It is commonly accepted that emotions affect learning, but emotions are more difficult to measure and study. According to Jensen (1998), three discoveries in the field of emotions have elevated the importance of studying the impact of emotions on learning: discoveries about the pathways and priorities of emotions in the brain; findings about the relationship between emotions and chemical secretions triggered by the brain; and the impact of emotions’ neurological priorities and biochemicals on learning and memory.

McLeod (1988) proposed that the affective characteristics of learning be subdivided into affect, emotion and attitude. Affect represents all of the feelings that relate to mathematics learning. Emotion describes the intense but short-lived sensations experienced in the moment. Attitude represents the relatively consistent, longer-term feelings about mathematics learning. Emotions create specific mind-body states, which create urges to respond or act. Key characteristics of emotions are their intensity, direction and duration. Students vary in their level of awareness of their emotional state. According to McLeod, “if problem solvers become aware of their emotional reactions, they may improve their ability to control their automatic responses to problems” (p. 137). Students also vary in their level of control over their emotional state. Past experience and
attitude are key factors in students’ conscious responses to their emotional state. Students who know that struggle is part of the problem solving process and have found success in the past after persisting, will probably calm themselves and persist again, knowing that finding the solution will be all the more satisfying given the effort to get there. Students without a track record of success will find it hard to summon the energy to try what they view as a hopeless task. Issues such as learned helplessness, math anxiety, and causal attributions of success and failure all feed into students’ attitudes about mathematics and problem solving, making the affective issues of learning complex but essential for progress. As mentioned above, Fuys et al (1988) observed several affective issues that aligned with student learning progress, including their comfort with exploration in mathematics, their degree of confidence, persistence, independence as a learner, and their tendencies for impulsive versus reflective conclusions.

Given that a student’s affect is a key factor in learning, then the instructor’s challenge is to create environments that trigger positive affective responses. Small group instruction, levels of teacher direction and scaffolding, group work and differentiated tasks are all instructional options to consider. McLeod (1988) noted that children tend to become deeply engrossed in computer work and as a result, this instructional environment can have interesting effects on affective responses. McLeod also recommended explicit instruction on affective issues. This would help students to monitor their affect and to appreciate the normal flow of feelings while working on a task.
Effective Access to Previous Learning

Whether one subscribes to a constructivist or cognitive schema theory of learning, the key issue is that students need to organize new learning into a framework that enables them to make sense of the learning and access it appropriately. The cognitive schema theory represents a middle ground between the information-processing and strong constructivist theories of cognition (Derry, 1996). It describes stored human knowledge in terms of memory objects, mental models and cognitive fields. As described by Derry, memory object schemas represent the permanent results of learning that are stored in memory and thus constitute the population of all preconceptions that a student might use to interpret any event. Cognitive fields represent the situationally activated preconceptions that are likely to be called on during the mental modeling process…. Mental models are particular organizations of memory objects that constitute a specific event interpretation. (p. 169)

Memory objects may be as simple as individual facts, or as complex as recognized relationships or patterns among collections of objects. It is essential that the memory objects are firmly implanted and interconnected in students’ cognition so that they are accessible when needed.

As stated above, geometry entails a greater emphasis on the learning and recall of vocabulary, definitions, postulates and previously demonstrated theorems. Usisken and Senk (1982) had found that while van Hiele level correlated to proof test results, the better predictor was the score on the content-based test. Lawson and Chinnappan’s study (2000) supported the statement that more extensive and better-structured geometric
knowledge positively correlates with better geometric problem-solving performance. They recommended that teachers devote class time for students to develop their own schema for organizing their mathematical knowledge.

Given the differences in the nature of geometry as compared to previous experiences with math, teachers and students need to explicitly address the changes in learning processes that are needed to absorb the new types of memory objects that geometry presents. What adds to the staying power and impact of a memory object?

Eric Jensen (1998) contended that formation of a lasting memory is influenced by emotion, relevance, context and patterns. “Emotions drive the threesome of attention, meaning and memory” (p. 94). One can purposefully engage emotions through positive experiences such as novelty of experience, multisensory experiences, working with others or personal investment. Relevance is the need-to-know component. It is the attachment of meaning to information, most often by connecting it through its similarities or differences to prior learning or experiences. Jensen stated that the “desire to form some kind of meaningful pattern out of learning seems innate” (p. 95) but “patterns can be forged and constructed only when enough essential ‘base’ information is already known” (p. 96). While that base is being built, the learning experiences need to be experiential and relevant for patterns to develop.

These patterns form the framework of the mental models that create the staying power and impact of new learning. The patterns of geometry are in some ways different from the patterns of previous experience in algebra. Some of the author’s students have commented that they found the procedural tasks of algebra easy to absorb but they find it much more difficult to recall the information they are learning in geometry. This may
imply comfort with numbers and sequential thinking but not with the less sequential nature of the visual/spatial realm. Teachers can explicitly assist students in translating patterns of all types, whether numeric, visual, linguistic or logical, into mental models through sharing of their own mental models, through graphic organizers, and through activities that require students to explain or compare and contrast sets of information (Jenson, 1998).

The Role of Dynamic Geometry Software in Learning Geometry

Dynamic geometry software can be a means to provide students with the experiential learning that enables them to see the geometric patterns emerge. It can connect to the emotional component of learning through multisensory experience, personal investment and immediate feedback. Instead of considering perhaps a handful of static examples, students can explore an almost endless continuum of cases within moments. This allows them to recognize patterns, form conjectures and test / refine their conjectures rapidly. Learning geometry in this mode presents differences from the static examples made from compass and straightedge constructions. The emphasis on visual observations skills is much greater in a dynamic setting than with static examples (Scher, 2002).

De Villiers (1998) proposed that the advent of the computer has made experimentation in many areas of mathematics more feasible, and as a result, has changed the role of deductive proof in mathematics. In the past, “the function (or purpose) of proof is seen as only that of the verification (conviction or justification) of the correctness of mathematical statements” (p. 370). Especially in the high school geometry class, once students have personally convinced themselves of a conclusion through the use of
dynamic geometry tools, the need to verify with deductive proof is not viewed as necessary. De Villiers has found that students find value in deductive proof as a means of understanding and explaining why a certain result occurs. “…when proof is seen as explanation, substantial improvement in students’ attitudes toward proof appears to occur (p. 388).

Olive (1998) outlined several educational implications of Geometer Sketchpad in the geometry classroom. The traditional approach of building up geometry from axioms, definitions and theorems is not appropriate when phenomena can be explored real-time. Inductive reasoning should be the focus, relying on experimentation, observation and conjecturing. He agreed with de Villiers that proof becomes more appropriately used for explanation than for verification. This approach allows students to construct mathematical relationships and meaning for themselves – which constructivists believe is the only way that learning is accomplished. Olive stated, “If used in conjunction with practical, physical experiences (such as ruler and compass constructions on paper), the computer construction tool can provide a link between the physical experiences and the mental representations” (p. 400). This statement aligns with the AIMS model of developing the interaction between concrete and representation. This also implies that dynamic geometry software aids in the development of students’ spatial visualization abilities.

There are many ways in which dynamic geometry software can be implemented in a classroom. Students might use the software to create the geometric objects themselves and then explore them around a certain goal. Or, dynamic examples can be prepared in advance for the students to use, with some structure around the questions to
be explored and answered. Drawings can be prepared that include measurements of
lengths and angles, enabling students to conjecture on numerical formulas. Stated
theorems can be captured in a drawing and explored to explain why they hold true. The
possibilities depend on the teacher’s expertise with the software, the access to
technological resources and the time available to invest in student learning of the
software.

Certainly having students create their own geometric objects requires a greater
classroom time investment and requires more independence and ownership on the part of
the student. Goldenburg and Cuoco (1998) described how geometry within DGS differs
from geometry on paper.

In fact, except for what students do in their heads, paper-and-pencil
gometry also ‘involves only action and does not require a description.’
Part of what students learn in geometry is, as Poincaré put it, the art of
applying good reasoning to bad drawings – adding the descriptions that
specify which features or relationships in the drawings are intended, which
are incidental, and which are to be totally ignored as they attempt to draw
inferences about the figure depicted. (p. 365)

Students who can successfully make this transition have demonstrated their mastery of
the third van Hiele level of abstractions: seeing the relationship between properties and
figures.

Geometer Sketchpad provides a great deal of flexibility on the personal computer
platform and appears to be the most prominent software tool. Another currently available
option is JavaSketchpad, offered by the makers of Geometer Sketchpad. According to
the Geometer Sketchpad Resource Center website (n.d.), *JavaSketchpad* is technology for viewing and interacting with dynamic visualizations created by someone else. They can be accessed via the Internet, or in some cases, downloaded onto an individual computer. Java Sketches are simple to use. Users just click on the illustrations and drag them about. These existing models can help jumpstart the integration of dynamic geometry software into the classroom through less teacher investment in material development, virtually no learning curve for students, and robust models that should minimize technology mishaps.

**New York State Regents Geometry Course**

In the 2008-2009 academic year, the new NYS Regents Mathematics core curriculum standards for Geometry were put into effect. With this change came the opportunity for teachers to revisit the objectives, priorities and approaches to teaching and learning geometry at the high school level.

The New York State Education Department’s Specifications for the Regents Examination in Geometry document provided relative weightings to the initial exam’s testing distribution among the five content strands in the standards (see figure 2). The algebra skills developed in Integrated Algebra are to be maintained through use in various geometry tasks. Although the weighting indicates that 41 – 47% of the exam pertains to informal and formal proof, careful reading of the Mathematics Core Curriculum document revealed that the dimensions of the proof content strand involve inductive as well as deductive reasoning, as well as application of the various theorems. The methods of acceptable proof are open-ended. The intent of the process and content performance indicators is to provide a variety of ways for students to acquire and demonstrate mathematical reasoning ability when solving problems. In summary,
Figure 2: Specifications for the Regents Examination in Geometry.
New York State Education Department. (n.d.).

The University of the State of New York
THE STATE EDUCATION DEPARTMENT
Albany, New York 12234

Specifications for the Regents Examination in Geometry
(First Administration—June 2009)

The questions on the Regents Examination in Geometry will assess both the content and the process strands of New York State Mathematics Standard 3. Each question will be aligned to one content performance indicator but will also be aligned to one or more process performance indicators, as appropriate for the concepts embodied in the task. As a result of the alignment to both content and process strands, the examination will assess students’ conceptual understanding, procedural fluency, and problem-solving abilities rather than assessing knowledge of isolated skills and facts.

There will be 38 questions on the Regents Examination in Geometry. The table below shows the percentage of total credits that will be aligned with each content band.

<table>
<thead>
<tr>
<th>Content Band</th>
<th>% of Total Credits</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometric Relationships</td>
<td>8—12%</td>
</tr>
<tr>
<td>Constructions</td>
<td>3—7%</td>
</tr>
<tr>
<td>Locus</td>
<td>4—8%</td>
</tr>
<tr>
<td>Informal and Formal Proofs</td>
<td>41—47%</td>
</tr>
<tr>
<td>Transformational Geometry</td>
<td>8—13%</td>
</tr>
<tr>
<td>Coordinate Geometry</td>
<td>23—28%</td>
</tr>
</tbody>
</table>

Question Types

The Regents Examination in Geometry will include the following types and numbers of questions:

<table>
<thead>
<tr>
<th>Question Type</th>
<th>Number of Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiple choice (2 credits each)</td>
<td>28</td>
</tr>
<tr>
<td>2-credit open ended</td>
<td>6</td>
</tr>
<tr>
<td>4-credit open ended</td>
<td>3</td>
</tr>
<tr>
<td>6-credit open ended</td>
<td>1</td>
</tr>
<tr>
<td>Total credits</td>
<td>86</td>
</tr>
</tbody>
</table>
students are expected to achieve a van Hiele level 4, with as much latitude as possible for expressing their logical thinking. The five skills of visual, drawing, verbal, logistic and application that Hoffer emphasized are also clearly represented in the standards.

In New York State, three math credits are a graduation requirement for all students receiving a Regents diploma (2007). Therefore the full spectrum of the student population enters the Geometry course. Students typically take Regents Geometry in 10th grade, at an average age of 15, after successfully completing the Regents Integrated Algebra course.

Students enter geometry with significant prior geometry classroom experiences. Geometry is a component of the NYS Mathematics Core Curriculum standards for grades 3 – 9 (see Appendix A). The individual standards indicate that students are exposed to learning experiences that should be enable them to recognize and name properties of geometric figures, placing them at the start of van Heile’s level 2 at the beginning of the geometry course.

Summary

The New York geometry course standards imply that the expectation still remains that students will need to progress from a van Hiele level 2 to a van Hiele level 4 over the duration of the Geometry course. This is not unlike the challenge that Usiskin and Senk described in 1982, with disheartening results for success in proof. It is clear from their study that the traditional Euclidean approach to geometry has not been sufficiently successful. The frustrations felt by students and teachers are understandable in light of the van Hiele theory. The axiomatic structure and deductive reasoning focus of Euclidean geometry represent level 4 understandings. When geometry teachers design
instruction from their vantage point of level 4 or 5 understandings, students at level 2 are not in a position to make sense of the instruction. It is too great of a leap, so students can only hope to memorize without conceptual understanding, in an attempt to get by.

The van Hieles recommended a different approach to instruction, based on a developmental perspective and inquiry targeted at reaching the next level of understanding. Students work through cycles of discovery, articulation, application and integration. Ongoing assessment of student progress is necessary to select the appropriate instructional experiences and teacher interventions. Hoffer (1981) stressed that geometry is defined by more than just deductive proof. He advocated that visual, drawing, verbal, logical and applied skills are all essential and need balanced development. Higher level tasks such as proof and problem solving require an integration of these component skills. The human brain is wired to seek patterns in new information, and this is a key way to create and retain memory objects (Jensen, 1998) Fuys, Geddes and Tischler (1988) documented the effectiveness of concrete manipulatives in aiding students to explore geometric patterns and properties. They also described the impact of student learning characteristics on learning progress. These affective factors are another critical component to address in the design of learning experiences. Dynamic geometry software offers promise as a tool to reinvent many aspects of geometry instruction, accelerating and deepening the process of exploration in an engaging manner. Without a doubt, there are many challenges to teaching geometry, but better student success is most certainly achievable.
Methodology

As the reviewed literature indicates, there are many dimensions and challenges to successfully nurturing the learning of geometry in high school. Optimally designed learning experiences maximize the students’ comprehension of new geometry knowledge, techniques and concepts. Learning experiences need to be experiential and relevant for patterns to develop and lasting memory objects to be created. While affective learning states are in large part an individual characteristic, the design of the learning experiences can swing a student’s emotional state in a positive or negative direction.

Hoffer’s enhancements to the van Hiele learning framework capture both the breadth of skills and sequential learning stages in a way that is well aligned to this researcher’s classroom observations. The results of research on the response of students to learning experiences involving dynamic geometry show great promise. The students in this researcher’s classroom had not had much exposure to dynamic geometry software, so there was curiosity as to how they respond to learning in that environment.

Building from the published literature, this study had two primary questions:

1) How does a unit of study, based on the van Hiele and other researchers’ recommendations of stages of learning, dimensions of skill involvement, differentiation and interaction, impact student retention of content and ability to apply the content in proof and situational problems?

2) How do students respond to learning experiences involving dynamic geometry software – in terms of engagement in the learning activity but also retention and full understanding of the concepts being studied?
Participants

The participants in this study were the two sections of Regents Geometry taught by the author at a public suburban high school. The suburb is moderately racially and socio-economically diverse for the Rochester area. At this high school, Geometry is offered at two levels: an advanced course taken by 25% of the students, and Regents Geometry, taken by the rest of the student population.

The Geometry sections were representative of the overall school population in terms of race and socio-economic status. Section 1 consisted of 23 students; fifteen boys and eight girls; two Black, three Hispanic, 17 White and one of Other race; 21 in tenth grade, two in eleventh grade (one repeating Geometry; the other had repeated Algebra). Section 2 consisted of 20 students; ten boys and ten girls; four Black, one Hispanic, 14 White and one of Other race; eighteen in tenth grade, two in eleventh grade (both repeating the course). Each Geometry section met daily for 42-minute class periods, the first and second class periods of the day.

The author’s two Geometry sections from the previous year were used for comparison purposes. Section A consisted of 21 students; thirteen boys and eight girls; two Black, three Hispanic, fifteen White and one of Other race; nineteen in tenth grade, two in eleventh grade (one repeating Geometry; the other had repeated Algebra). Section B consisted of 22 students; nine boys and thirteen girls; four Black, three Hispanic, fourteen White and one of Other race; twenty in tenth grade, two in eleventh grade (one repeating Geometry; the other had repeated Algebra). These Geometry sections also met daily for 42-minute class periods, the first and last periods of the day.
Design and Procedure

The quadrilaterals unit was the focus of this study. The unit was twenty-one days long, straddling the New Year vacation. The high school team of geometry teachers defined the content of the unit (appendix B) during summer work in 2008, building off the New York State Education Department’s Math Core Curriculum for Geometry (2005). Appendix C captures the daily objectives for the unit along with the author’s assessment of the linkages to the van Hiele instructional phases and Hoffer’s skill breakdown.

In overview, the unit began with a recall of prior learning about the quadrilaterals and their most basic properties and definitions. The relationships or hierarchy among quadrilaterals were surfaced using the words always and sometimes. Properties pertaining to the diagonals of quadrilaterals were then explored using Java Sketches and students were guided into making conjectures about the properties. These properties were reinforced through practice in applying the properties to determine segment and angle measures in diagrams. This was followed by students summarizing and organizing the set of properties in both hierarchical and matrix graphic organizer structures. The idea of a minimum but sufficient set of properties to uniquely identify a quadrilateral was developed through a game format similar in nature to the children’s game, Guess Who. Sufficient sets of properties for each quadrilateral type were next determined for use in coordinate geometry proof. Strong emphasis was placed on the components of the logical argument presented in the conclusions. The parallelogram family quiz on Day 10 assessed their mastery of the properties and the interrelationships between the types.
Upon return from the holiday, application problems involving quadrilaterals were reintroduced, where students needed to translate descriptions into diagrams, or transfer information onto diagrams, and then recall the relevant properties to solve the problems. Finding areas and perimeters of quadrilaterals was taught in Integrated Algebra and prior years, but was refreshed and strengthened in light of the quadrilateral properties. The interior and exterior angle patterns of triangles and quadrilaterals were then expanded into pentagons, hexagons and beyond. On Day 19, deductive reasoning was brought in as a way to explain why the diagonal properties are true. Students applied their experience in the prior unit with congruent triangle proofs to prove the diagonal properties of various quadrilaterals.

In terms of van Hiele levels, days 1 through 5 of the unit consolidated previous learning and added new quadrilateral properties. This work placed students at the completion of level 2, Analysis and the beginning of level 3, Abstraction. The concept of sufficient sets of properties is a key component of level 3. This was developed and applied in days 6 through 9 in the context of coordinate geometry proofs. The day 10 quiz marked the end of the focus on knowing. After the winter break the focus transitioned into the higher level thinking of applying the knowledge in problem-solving. Level 4, Deduction was brought in on day 19 in a very scaffolded format, with opportunities for students to extend into writing their own proofs.

When areas and perimeters were brought in, the nature of the work stepped back to an earlier instructional stage as students recalled and expanded their understandings of area and perimeter. The same instructional process occurred when polygons were
introduced, stepping back to the Information phase as students made sense of this new collection of objects.

As a key focus of this study was the student response to working with dynamic geometry software, these sessions were documented in this report in greater detail. Students worked in the school’s computer lab, accessing Java Sketches available on Key Curriculum Press’s Discovering Geometry textbook support website (2009). While students each had their own computer and worksheets, they also were encouraged to collaborate with their neighbors to resolve questions as they arose. Student worksheets closely followed the flow of the website design. In order to answer the question, ‘How do students respond to working with dynamic geometry software’, a second math teacher collected observation data during each session regarding level of engagement and types of conversations between partners. The author also documented observations on the nature of student to student and teacher-student interactions while in the lab. Student worksheets were also collected to review the nature of their responses.

The students had their initial exposure to working in the computer lab with dynamic geometry software on the two days of the study. It was hoped that there could have been an earlier opportunity to get students adjusted to the novelty of working in the computer lab so that any initial startup issues could have been resolved, allowing the focus on days 2 and 3 to be strictly on the impact of the Java Sketches and associated worksheets. Unfortunately, that could not be arranged. Any effects of this first experience will have to be addressed in the analysis of the data collected.

The assessments on days 10 and 21 provided quantitative feedback on student progress. The day 10 quiz had similar objectives but a different format than that used in
the previous year, but the unit test on day 21 was nearly the same, enabling a comparison with the previous year’s results. The midterm exam questions on quadrilaterals also provided a source of comparisons with the previous year’s results.
Results

The overall impact of the quadrilateral unit’s instructional design on student understanding was measured qualitatively through teacher observations and quantitatively through the quiz and midterm assessments. The dynamic geometry software experiences took place on the second and third days of the unit (see Appendix C). The results will be discussed in chronological sequence.

There were significant differences in the design of the instructional activities for this year as compared to the previous year. Last year, the quadrilateral unit started out at a van Hiele level 4 with no data to support whether all students should begin instruction at this level. Properties were most often stated as facts, primarily in words with occasional diagram representations, and triangle proof techniques were used in some cases to justify the statements. Students were then expected to apply the properties to problem solving tasks. Visualization skills of the students were weak but were not explicitly practiced and developed. When the first major quiz made it clear that students could not recall the properties of quadrilaterals, more investment was made in learning the properties through connections between drawings of representative quadrilaterals and the property statements and the students were retested with a very similar quiz.

This year, as the Daily Objectives of Quadrilaterals Unit (Appendix C) describes, the unit was carefully structured to begin at van Hiele level 2, integrate a variety of skill types, and progress through the sequence of instructional stages. Figure 3 shows the first activity of the unit. In a “think, pair, share” context, students were asked to use the examples and non-examples to provide a definition of each shape. Students were actively participating but required significant teacher prompting and questioning to get to accurate
definitions of trapezoids and parallelograms. Squares and rectangles were then easier. Students had difficulty selecting the appropriate words to clearly describe the patterns they were observing in the shapes. For example, “straight” was used for “horizontal” and in some cases for “parallel”. For some students, the changing orientations of the shapes affected their ability to see the common characteristics between them. Angles were right only when the sides were vertical and horizontal. One of the key characteristics of the van Hiele level 3 of abstraction is the ability to recognize the class structure of quadrilaterals. In order to gather information on students’ current understanding of class structures, students were asked to individually answer the questions in Figure 4. Their results on each question were then informally documented by voting for never, sometimes or always. The majority of students had some level of incorrect answers. The most common issue was to answer never instead of sometimes. Students were highly engaged in the group discussion that focused on carefully rereading and interpreting the definitions of each quadrilateral. One by one, students came to understand the hierarchical structure, which was then captured at the bottom of the page.

Days 2, 3 and 4 were invested in student exploration, discovery and description of the properties of quadrilaterals before coordinate geometry proof began. On Days 2 and 3, students used dynamic geometry software in the computer lab, while on Day 4, students did hands-on measurement of static examples in the classroom.

As the student response to dynamic geometry software was one of the main research questions, the student engagement in the dynamic geometry software sessions was measured in multiple ways. The observing math teacher provided a 1-10 numerical rating of each student based on his observations of student focus and on-task actions.
Figure 3: *Creating Definitions of Special Quadrilaterals*

These are trapezoids:

These are not trapezoids:

These are parallelograms:

These are not parallelograms:
Figure 4: Assessing Class Structure Knowledge of Quadrilaterals

Determine whether each statement is true ALWAYS, SOMETIMES or NEVER. Explain your reasoning.

1. A rectangle is a square.
2. A square is a rectangle.

3. A rhombus is a square.
4. A square is a rhombus.

5. A rectangle is a parallelogram.
6. A parallelogram is a rectangle.

7. A trapezoid is a parallelogram.
8. A parallelogram is a trapezoid.

“Family Tree” of Quadrilaterals

________________

________________   __________________

________________   __________________   __________________

________________
For comparison, the author provided a rating of each student’s average level of focus and on-task actions during routine classroom lessons based on classroom observations. This data has been combined into Table 1 for all attending students. A T-test of Day 1 versus the Average Classroom Day showed a significant difference in the means (p < .05). A T-test of Day 1 versus Day 2 also showed a significant difference in the means. A T-test of Day 2 versus the Average Classroom Day showed that the means were not significantly different.

Both the observing teacher and the author recorded observations about student and teacher interactions during the lab sessions. Students learned quickly how to manipulate the Javasketch images on the webpages. Those students with lower reading levels had more difficulty in following the instructions provided on the lab sheets. The author checked in more frequently to assist these students in coaching them along to the next step. While the students each worked at their own computer with their own lab sheets, many chose to discuss and validate their observations with their neighbors as they worked. Most students handled the freedom of computer access responsibly, while a few needed teacher reminders of consequences for off-task behaviors. The number of behavior issues increased on Day 2 in the lab, which correlates with the decrease in student engagement.

On Day 5, students provided written reflections to the prompt, “Over the past three days, you have explored properties of different types of quadrilaterals. You spent two days in the computer lab working with computer images and one day in class using a protractor and ruler on examples on paper. Thinking about yourself and how you learn best, which do you prefer and why? What do you think are the strengths and weaknesses
Table 1: *Computer Lab Student Engagement Ratings*

<table>
<thead>
<tr>
<th>Student Identifier</th>
<th>Average Classroom Day</th>
<th>Computer Lab Day 1</th>
<th>Computer Lab Day 2</th>
</tr>
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</tr>
<tr>
<td>34</td>
<td>4</td>
<td>absent</td>
<td>5</td>
</tr>
</tbody>
</table>

Average Rating: **7.03**  **8.12**  **7.41**
of each approach?” Of the 30 students who provided feedback, nine preferred the classroom over the computer lab, 20 liked the computer lab experience and one was content with either environment.

Student work from the lab session and the subsequent homework was collected for analysis of task completion and understanding. Students were to discover the patterns and describe the properties in words and in diagrams. Successful description of the property was assigned a value of 1; unanswered or unsuccessful descriptions were assigned a value of 0. The related homework assignments emphasized numerical calculations and restatement of the properties that were applied. Successful completion scores were assigned for both the numerical and restatement portions of the homework. Each student’s worksheet and homework was evaluated and the averaged results are displayed in Table 2. There is a large difference between the percent successfully describing of the property in words versus description in the diagram on both of the computer lab days. For the most part, students did not attempt to complete the diagram, despite being reminded of the importance to do so during the lab time. There is also a significant drop in successful completion between Day 1 and Day 2. This correlates with the drop in student engagement level that was observed.

Day 6 began by students re-sorting the quadrilateral properties from a hierarchical format into a matrix organizer. The idea of sufficient properties to uniquely identify a quadrilateral was introduced through connection to the popular game, Guess Who, which had been played by over three-quarters of the students. Those who had not played the game quickly understood it after a brief explanation and informal round of the game. The matrix form of the organizer was then displayed while students asked questions to
<table>
<thead>
<tr>
<th>Property</th>
<th>Description</th>
<th>Correct Response Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Day 1 - Parallelograms</strong></td>
<td></td>
<td></td>
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<tr>
<td>Opposite angles are congruent</td>
<td>Words</td>
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</tr>
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<td></td>
<td>Diagram</td>
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<tr>
<td>Consecutive angles are supplementary</td>
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<td></td>
<td>Diagram</td>
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<td>Opposite sides are congruent</td>
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<td></td>
<td>Diagram</td>
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<tr>
<td>Diagonals bisect each other</td>
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</tr>
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<td></td>
<td>Explanation</td>
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<td><strong>Day 2 - Special Parallelograms</strong></td>
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<td>Rhombus: diagonals are perpendicular</td>
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<td></td>
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<td>Explanation</td>
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identify the secret quadrilateral in as few questions as possible. Students were enthusiastic and actively participated in the game. Over the course of four rounds, the class became proficient at asking a minimum number of characteristic questions to uniquely identify the quadrilateral. The concept was then extended into coordinate geometry proof, where measurable characteristics demonstrate properties, which in turn uniquely define a quadrilateral type.

Working on coordinate geometry proofs through Day 9 provided the students with multiple opportunities to work with the various properties, with the intention of building strong memory objects. Additionally, this format of proof is more concrete and procedural, providing students with the opportunity to develop their logical thinking skills. Warm-up quizzes were given on days 7, 8 and 9 to motivate and assess student mastery of the previous day’s quadrilateral properties and the coordinate proof method. The day 7 warm-up highlighted that many students did not understand that measurable characteristics such as slope or distance are used prove the presence of a quadrilateral properties and that both the measurements and the property must be stated.

On Day 10’s quiz, students demonstrated their recall of the quadrilateral properties by choosing and completing one of the graphic organizers that had been used in class. They also answered multiple-choice problems involving the properties and completed two coordinate geometry proofs. Students appeared confident and engaged as they worked on the quiz. Their results showed that knowledge of the properties was generally high, but students were less successful in selecting the correct answers on multiple-choice questions targeting the quadrilateral properties, and there was considerable variation in the degree of mastery of the proofs. Table 3 shows the quiz
Table 3: Comparison of Quadrilateral Unit Assessment Results

<table>
<thead>
<tr>
<th>Day 10 Quiz Study Year</th>
<th>Day 8 Quiz Previous Year</th>
<th>Unit Test Study Year</th>
<th>Unit Test Previous Year</th>
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Mean 80% 63% 75% 68%
results for the study year as well as the results of the first quiz for the previous year. The quiz mean for the study year was 17 percentage points higher than the quiz mean for the previous year. A T-test shows that the difference in the means between the two quizzes is significant (p < .05). While the content being tested was the same, there were some differences between the two quizzes, most notably the format and skill type of some of the questions and the timing of the quiz within the unit.

Returning from the holiday break, the Day 14 objective was for students to recall the quadrilateral properties as they design an application problem based on one or more properties. A subset of the student-designed problems was later integrated into the unit study guide that was the focus of Day 20. Many students struggled with this task. They understood the property they had chosen, but the source of the difficulty seemed to be more with how to design an application problem. Days 15 – 18 addressed a variety of topics that best fit in the quadrilaterals unit, but the recall and application of quadrilateral properties was a necessary component in many situations, giving students additional opportunities to recognize and draw on the knowledge they had gained.

Day 19 was devoted to experiences aimed at enabling the students to apply deductive reasoning processes in the form of traditional statement and reason proofs to explain why the observed quadrilateral properties follow from the specific quadrilateral definition. Based on the research indicating the broad range of readiness to be able to make this jump, this lesson was purposefully designed to be scaffolded to ensure that all students would achieve some level of success. The classwork (Appendix D) consisted of four partially completed proofs, one framed proof, and three properties that could be proven. The first proof was completed as a whole class to model the process and then
students worked at their own pace with a partner. Most students were able to fill in the missing parts in the first four proofs, one-fifth attempted the framed proof with varying degrees of success, and only one student attempted the final task.

The unit concluded with a unit test on Day 21. Test results are presented in Table 3. A comparison is made to the unit test from the previous year. While the mean of the study year test is 75% compared to 68% for the previous year, a T-test indicates that this difference in means is not significant. The comparison between the two tests is not ideal, as there were significant differences in the content of the two tests. In the previous year, the unit test was in large part a retest due to the poor performance on the prior quiz. In the study year, the unit test was more comprehensive and challenging.

The same questions pertaining to quadrilaterals were asked on the midterm for the study year and the previous year. There were five multiple choice questions and three extended response questions. Only aggregate class averages are available for the previous year, so a T-test of significance could not be completed. The average multiple choice score on the quadrilateral questions was 67.4% for the study year compared to 63.7% for the previous year. The average extended response score on the quadrilateral questions was 68.1% for the study year compared to 64.0% for the previous year. In aggregate, the average score on the quadrilateral questions was 67.9% for the study year compared to 63.9% for the previous year.
Discussion

The first research question was: How does a unit of study, based on the van Hiele and other researchers’ recommendations of stages of learning, dimensions of skill involvement, differentiation and interaction, impact student retention of content and ability to apply the content in proof and situational problems?

The summative assessment results for this quadrilateral unit are somewhat inconclusive in that they show only very slight increases in mean scores for the study year as compared to the previous year. Some of these mean differences are not statistically significant.

However, there were many moments within this redesigned unit when it was apparent that students experienced significant learning events in ways that the author had not witnessed or fully appreciated before. Misconceptions were surfaced and resolved. Gaps in various skills were diagnosed, thereby enabling focused interventions to begin. Even if these events didn’t dramatically impact the assessments that were in place for the unit, they definitely enhanced student understanding and learning progress.

On Day 1 of the unit, the author was concerned that the “think, pair, share” activity of creating definitions of various quadrilaterals based on examples and non-examples would be too simple for high school students. The class discussion, however, highlighted the students’ need to be assisted with appropriate geometric vocabulary to describe the patterns they were observing. This affirmed the van Hiele instructional phase of explicitation. The never, sometimes or always statements regarding relationships between quadrilaterals served as a formative assessment that indicated that the majority of students were operating at a van Hiele level 2. A key aspect to the
success of this activity was in publicly recording the number of responses within the class for never, sometimes or always for a given statement. It led to a friendly sense of competition, more animated discussion, and increased student investment in wanting to learn the correct response and seeking to understand why it was true.

The use of the game, Guess Who, to introduce the concept of sufficient properties to uniquely identify a quadrilateral was very effective at creating a playful motivation for an efficient yet conclusive reasoning approach. It was intended that this connection would contribute toward building a positive emotional/affective response toward the task of deductive proof while enabling students to understand the concept of sufficiency. The level of energy in student participation and the effectiveness of their questioning in Guess the Quad indicated that these intentions were realized.

The scaffolded design of the Day 19 application of deductive reasoning to explain quadrilateral properties in the form of traditional statement and reason proofs was greatly influenced by Usiskin & Senk’s statements that many students are unable to master deductive proof during their year of high school geometry. As stated earlier, most students were able to fill in the missing parts in the first four proofs, one-fifth attempted the framed proof with varying degrees of success, and only one student attempted the final task. Last year, the author did not anticipate the need for scaffolding and was surprised and discouraged when most students grew frustrated by their inability to make progress on the proofs. This year, the author was able to differentiate in advance so that all students could be appropriately challenged and make personal progress. It is questionable, however, as to whether students appreciated, as De Villiers (1998) proposed, that deductive proof is valuable to explain why the observed properties hold
true for all cases. The proofs were generally considered to be just another task to be completed, another way to practice using the definitions and theorems that had been learned previously.

In addition to the instructional changes, the van Hiele approach has greatly impacted the author as a teacher. I was very conscious of how the framework impacted my interactions with the students. I was constantly assessing students’ van Hiele level and proficiencies in the five skill areas described by Hoffer. With this assessment in mind, I was able to tailor my questions and feedback to fit what I anticipated to be the student’s next developmental steps. This approach has increased the quality of my interactions with students and given me insight into the learning progression of the students. It has enabled me to anticipate the needs of students in terms of developmental readiness, skill categories and affective characteristics, and incorporate those needs into differentiated lesson designs.

There may be some reasons why the formal assessments did not reflect the same level of progress that occurred in many of the lessons. While the students may have discovered the quadrilateral properties effectively, they may not have had adequate practice with or organization of those properties to build long term retention of the knowledge. The number of repetitions needed to retain information varies greatly by individual. This may explain why the average score on the quiz was 80% but the unit test average dropped to 75% and the midterm questions on quadrilaterals dropped to an average of 68%. It would be interesting to look for a correlation between test grades and homework completion rates. There may also be different sorts of classroom and
homework exercises that are more effective at helping students organize and retain their newly gained knowledge.

Another possible factor may be the ability of the student to interpret or apply the quadrilateral property knowledge in visual, drawing, verbal, logical and/or applied forms. Future study could involve analyzing student performances by skill type to diagnose areas of strength versus areas to target additional development.

The second research question was: How do students respond to learning experiences involving dynamic geometry software, in terms of engagement in the learning activity but also retention and full understanding of the concepts being studied?

The two days of exploration with dynamic geometry software proved to be more engaging and effective for the majority of students, based on observations of their behaviors (Table 1) and their own written feedback. This result is aligned with McLeod’s finding (1988) that students tend to become deeply engrossed in computer work. One student commented, “I think the computer lab is a lot better to work in because it allows a visual shape that you can interact with and change to help you understand key concepts. And I’m a visual learner so it really helps.” Another stated, “I think that working in the lab helped a lot because it provided as many visuals as you could think of. It allows the student to play with the shapes and see that all of the properties really do apply no matter what.” These and other similar comments confirm the findings of Scher (2002) and Olive (1998) that the work with dynamic geometry software emphasizes inductive thinking and visual observation skills, and allows the students to construct meaning for themselves.
Of concern, however, is the drop in student engagement on the second day in the computer lab, combined with the lower rate of success at recording the observed patterns in the worksheet. In order to assess the strengths as well as weaknesses of the lab experiences, it will be useful to look at four factors impacting the results of the experience: the Java Sketches on the computer screen, the associated lab sheets, the arrangement of the students, and the role of the teacher.

On the first day in the lab, it was a fresh and novel experience. There were enough computers for each student to have their own, and they sat in close proximity to each other so consulting with a neighbor was easy to do. Students were allowed to choose their own seats, and for the most part, friends sat next to each other. It took the students some time to get used to reading the instructions on the lab sheet and interacting with the Java Sketches, but neighboring students helped each other and the patterns were easy to find and describe. Most were done before the period was over and they had started the attached homework. There were very few management issues. On the second day in the lab, students came in, sat where they liked and got moving quickly, but then there were more behavior management issues, more questions and more frustrations. As one student commented in his feedback, “I did not like some parts because it was confusing on how to answer the problems.” Another described, “In the computer lab you aren't actually sure if you are getting the right answers but when you're in class you get the answers.” As the analysis of student work supports, students were indeed not arriving at the correct conclusions to the extent that they did on the first day. Upon reflection, the root cause of this appears to be that the patterns to be discovered were new to students,
they may have been more difficult to recognize in the Java Sketches, and the lab sheets may have been less clear as to what was to be captured.

The role of teaching in the computer lab was much like a workshop environment in the classroom. After providing initial instructions, the teacher circulated, monitoring students’ progress and providing individual coaching as needed. Students worked at their own pace and recorded their work in the lab sheets. Several of the students with attention issues found the computer environment helpful. “I liked the computer lab a lot better than being in the classroom for a few different reasons. 1) all the work and materials are all on one screen; you don't have a ton of papers and materials to worry about. It makes things easier to concentrate. 2) There's always someone to get help from. The teacher has more time to help with questions.” More time was available to assist individual students, but it was important to make careful choices with the time. Some students tried to rely on the teacher instead of reading and thinking for themselves, while others needed teacher intervention because they were making wrong conclusions without realizing it.

By the end of each class period, students were at different points within the discovery and the practice homework. While the majority in each class had finished the discovery and accurately captured the patterns, some had rushed through the work and made incorrect conclusions. Some who had struggled with the reading and writing aspects of the assignment and had not yet completed the lesson portion. These issues happened more on the second day than on the first. The environment of the computer lab was such that whole group presentation was difficult, so clarifications and closure of the day’s objectives were not sufficiently effective.
After three days of discovery, there was a large amount of new information that students needed to organize into an effective mental model. In addition, successful homework completion rates in the sixty percent range (see Table 2) indicated that student success on each day’s homework tasks was lower than their performance in the lab. A few did not attempt the homework, but for the majority who did, it was clear that they were not making the connection between the properties they discovered and the application of those properties in problem solving. Perhaps too much new information had been introduced without allowing time for the van Hiele instructional phases of free orientation and integration.

There are several alternatives for changes that could alleviate student anxieties and reinforce correct results before application and practice are begun. As students completed the tasks, they could check their work against an answer key. If the lab were set up with a centrally visible display, the Java Sketch and lab sheet could be displayed with correct conclusions at the end of each session. Another option is to schedule a classroom day between the computer lab sessions to confirm, practice and apply the first day’s learning and also set the stage for the subsequent day’s tasks.

Another consideration is the optimal design of the Java Sketch and its relationship to the lab sheets. Publicly available Java Sketches were used and the lab sheets were designed to guide the students through the use of these Java Sketches. The webpages on which these Javasketches were located were also filled with verbage to be ignored. It was observed that the three-letter angle naming conventions were difficult for some students to follow. Simpler Java Sketches could be created and the lab sheets could
include snapshots of what is seen on the screen to add clarity to the presentation of the tasks.

One of the objectives of the computer lab sessions was to develop the recognition and description of the quadrilateral properties in visual as well as verbal form on the lab sheets. When the need to access properties occurs in an application or a proof situation, the trigger is more typically presented in diagrammatic rather than verbal form. Analysis of the lab sheets revealed that far fewer students captured the properties in diagrammatic form than in written form. It was not clear why students did not take this step despite being reminded, but it should be analyzed in future investigations. This may have been a contributor to the low success rates in the homework assignments, as the problems required students to select the required property based on interpretation of relevant information in a diagram. Other instructional interventions should be considered to nurture students’ development of the visual interpretation skills.

Overall, the dynamic geometry experiences in the computer lab were effective additions to the mix of instruction methods. They provided a welcome change of scenery from the classroom and appealed to the majority of the students. Some modifications to the lesson designs should address the concerns of those students who did not prefer the lab environment.
Conclusion

The unit of instruction implemented in the author’s classroom was developed on the principles of the van Hiele model, Hoffer’s skill components, cognitive learning theories and brain research findings with the objective of improving student understanding. Learning methods included inferential thinking based on pencil and paper exploration of static examples or computer-based dynamic geometry explorations, practical applications, algebraic problem-solving and deductive proof. These experiences were chosen for their fit with the van Hiele instructional sequence recommendations and integrated the skill types described by Hoffer and the AIMS Education Foundation.

Observations of the students’ development, struggles and learning results were well aligned with the published research findings. Students were more engaged and more successful when given the opportunity to discover the quadrilateral properties for themselves and particularly enjoyed doing so with dynamic geometry software. Building strong memory objects required varying levels of practice among students. Multiple modes of presentation and extended exposure to the properties in varying contexts enabled a higher percentage of students to successfully recognize and access quadrilateral properties in problem solving and testing contexts.

Although the assessment result increases documented in this study were not statistically significant over the previous year, further enhancements could lead to even more significant student outcome improvements. In particular, the lab sheets and Java Sketches used for the discovery of quadrilateral properties could be more clearly designed. The opportunities to integrate dynamic geometry software extend far beyond individual experimentation in a computer lab. Java Sketches and other forms could be
integrated into quick whole-class demonstrations, group projects and extension opportunities.

Most importantly, the van Hiele model combined with Hoffer’s skill dimensions provided a framework for understanding the growth of student understanding in geometry, a mathematics subject substantially different from other high school math courses. With this framework in mind, teachers are better prepared to assess their students and develop differentiated instruction targeted at meeting their specific needs.
References


Van Hiele-based Geometry Instruction


Yazdani, M. (2007). Correlation between students’ level of understanding geometry according to the van Hieles’ model and students’ achievement in plane geometry. *Journal of Mathematical Sciences & Mathematics Education.*

Appendix A: Geometry Components of NYS Math Core Curriculum


http://www.emsc.nysed.gov/3-8/MathCore.pdf

Geometry Content Strands in:
Mathematics Core Curriculum MST Standard 3 Prekindergarten - Grade 12 Revised March 2005

Integrated Algebra Math core curriculum: Page 97-98

Geometry Strand

Students will use visualization and spatial reasoning to analyze characteristics and properties of geometric shapes.

Shapes

A.G.1 Find the area and/or perimeter of figures composed of polygons and circles or sectors of a circle Note: Figures may include triangles, rectangles, squares, parallelograms, rhombuses, trapezoids, circles, semi-circles, quarter-circles, and regular polygons (perimeter only).

A.G.2 Use formulas to calculate volume and surface area of rectangular solids and cylinders

Students will apply coordinate geometry to analyze problem solving situations.

Coordinate Geometry

A.G.3 Determine when a relation is a function, by examining ordered pairs and inspecting graphs of relations

A.G.4 Identify and graph linear, quadratic (parabolic), absolute value, and exponential functions

A.G.5 Investigate and generalize how changing the coefficients of a function affects its graph

A.G.6 Graph linear inequalities

A.G.7 Graph and solve systems of linear equations and inequalities with rational coefficients in two variables (See A.A.10)

A.G.8 Find the roots of a parabolic function graphically Note: Only quadratic equations with integral solutions.

A.G.9 Solve systems of linear and quadratic equations graphically Note: Only use systems of linear and quadratic equations that lead to solutions whose coordinates are integers.

A.G.10 Determine the vertex and axis of symmetry of a parabola, given its graph (See A.A.41) Note: The vertex will have an ordered pair of integers and the axis of symmetry will have an integral value.
8\textsuperscript{th} grade Math core curriculum: Page 85-87

**Geometry Strand**

*Students will use visualization and spatial reasoning to analyze characteristics and properties of geometric shapes.*

**Constructions**

8.G.0 Construct the following, using a straight edge and compass:
- Segment congruent to a segment
- Angle congruent to an angle
- Perpendicular bisector
- Angle bisector

**Students will identify and justify geometric relationships, formally and informally.**

**Geometric Relationships**

8.G.1 Identify pairs of vertical angles as congruent
8.G.2 Identify pairs of supplementary and complementary angles
8.G.3 Calculate the missing angle in a supplementary or complementary pair
8.G.4 Determine angle pair relationships when given two parallel lines cut by a transversal
8.G.5 Calculate the missing angle measurements when given two parallel lines cut by a transversal
8.G.6 Calculate the missing angle measurements when given two intersecting lines and an angle

**Students will apply transformations and symmetry to analyze problem solving situations.**

**Transformational Geometry**

8.G.7 Describe and identify transformations in the plane, using proper function notation (rotations, reflections, translations, and dilations)
8.G.8 Draw the image of a figure under rotations of 90 and 180 degrees
8.G.9 Draw the image of a figure under a reflection over a given line
8.G.10 Draw the image of a figure under a translation
8.G.11 Draw the image of a figure under a dilation
8.G.12 Identify the properties preserved and not preserved under a reflection, rotation, translation, and dilation

**Students will apply coordinate geometry to analyze problem solving situations.**

**Coordinate Geometry**

8.G.13 Determine the slope of a line from a graph and explain the meaning of slope as a constant rate of change
8.G.14 Determine the y-intercept of a line from a graph and be able to explain the y-intercept
8.G.15 Graph a line using a table of values
8.G.16 Determine the equation of a line given the slope and the y-intercept
8.G.17 Graph a line from an equation in slope-intercept form (y = mx + b)
8.G.18 Solve systems of equations graphically (only linear, integral solutions, y = mx + b format, no vertical/horizontal lines)
7th grade Math core curriculum: Page 76-77

Geometry Strand
Students will use visualization and spatial reasoning to analyze characteristics and properties of geometric shapes.

Shapes

7.G.1 Calculate the radius or diameter, given the circumference or area of a circle
7.G.2 Calculate the volume of prisms and cylinders, using a given formula and a calculator
7.G.3 Identify the two-dimensional shapes that make up the faces and bases of three-dimensional shapes (prisms, cylinders, cones, and pyramids)
7.G.4 Determine the surface area of prisms and cylinders, using a calculator and a variety of methods

Students will identify and justify geometric relationships, formally and informally.

Geometric Relationships

7.G.5 Identify the right angle, hypotenuse, and legs of a right triangle
7.G.6 Explore the relationship between the lengths of the three sides of a right triangle to develop the Pythagorean Theorem
7.G.7 Find a missing angle when given angles of a quadrilateral
7.G.8 Use the Pythagorean Theorem to determine the unknown length of a side of a right triangle
7.G.9 Determine whether a given triangle is a right triangle by applying the Pythagorean Theorem and using a calculator

6th grade Math core curriculum: Page 67

Geometry Strand
Students will use visualization and spatial reasoning to analyze characteristics and properties of geometric shapes.

Shapes

6.G.1 Calculate the length of corresponding sides of similar triangles, using proportional reasoning
6.G.2 Determine the area of triangles and quadrilaterals (squares, rectangles, rhombi, and trapezoids) and develop formulas
6.G.3 Use a variety of strategies to find the area of regular and irregular polygons
6.G.4 Determine the volume of rectangular prisms by counting cubes and develop the formula
6.G.5 Identify radius, diameter, chords and central angles of a circle
6.G.6 Understand the relationship between the diameter and radius of a circle
6.G.7 Determine the area and circumference of a circle, using the appropriate formula
6.G.8 Calculate the area of a sector of a circle, given the measure of a central angle and the radius of the circle
6.G.9 Understand the relationship between the circumference and the diameter of a circle

5th grade Math core curriculum: Page 57-58
*Students will recognize, use, and represent algebraically patterns, relations, and functions.*

Patterns, Relations, Functions
5.A.7 Create and explain patterns and algebraic relationships (e.g., 2, 4, 6, 8...) algebraically: 2n (doubling)
5.A.8 Create algebraic or geometric patterns using concrete objects or visual drawings (e.g., rotate and shade geometric shapes)

Geometry Strand
*Students will use visualization and spatial reasoning to analyze characteristics and properties of geometric shapes.*

Shapes
5.G.1 Calculate the perimeter of regular and irregular polygons

Geometric Relationships
5.G.2 Identify pairs of similar triangles
5.G.3 Identify the ratio of corresponding sides of similar triangles
5.G.4 Classify quadrilaterals by properties of their angles and sides
5.G.5 Know that the sum of the interior angles of a quadrilateral is 360 degrees
5.G.6 Classify triangles by properties of their angles and sides
5.G.7 Know that the sum of the interior angles of a triangle is 180 degrees
5.G.8 Find a missing angle when given two angles of a triangle
5.G.9 Identify pairs of congruent triangles
5.G.10 Identify corresponding parts of congruent triangles

Students will apply transformations and symmetry to analyze problem solving situations.
Transformational
5.G.11 Identify and draw lines of symmetry of basic geometric shapes

Students will apply coordinate geometry to analyze problem solving situations.
Coordinate Geometry
5.G.12 Identify and plot points in the first quadrant
5.G.13 Plot points to form basic geometric shapes (identify and classify)
5.G.14 Calculate perimeter of basic geometric shapes drawn on a coordinate plane (rectangles and shapes composed of
rectangles having sides with integer lengths and parallel to the axes)

4th grade Math core curriculum: Page 48

**Geometry Strand**

*Students will use visualization and spatial reasoning to analyze characteristics and properties of geometric shapes.*

**Shapes**

4.G.1 Identify and name polygons, recognizing that their names are related to the number of sides and angles (triangle, quadrilateral, pentagon, hexagon, and octagon)

4.G.2 Identify points and line segments when drawing a plane figure

4.G.3 Find perimeter of polygons by adding sides

4.G.4 Find the area of a rectangle by counting the number of squares needed to cover the rectangle

4.G.5 Define and identify vertices, faces, and edges of three-dimensional shapes

*Students will identify and justify geometric relationships, formally and informally.*

**Geometric Relationships**

4.G.6 Draw and identify intersecting, perpendicular, and parallel lines

4.G.7 Identify points and rays when drawing angles

4.G.8 Classify angles as acute, obtuse, right, and straight

3rd grade Math core curriculum: Page 39

**Geometry Strand**

*Students will use visualization and spatial reasoning to analyze characteristics and properties of geometric shapes.*

**Shapes**

3.G.1 Define and use correct terminology when referring to shapes (circle, triangle, square, rectangle, rhombus, trapezoid, and hexagon)

3.G.2 Identify congruent and similar figures

3.G.3 Name, describe, compare, and sort three-dimensional shapes: cube, cylinder, sphere, prism, and cone

3.G.4 Identify the faces on a three-dimensional shape as two-dimensional shapes

*Students will apply transformations and symmetry to analyze problem solving situations.*

**Transformational Geometry**

3.G.5 Identify and construct lines of symmetry
2nd grade Math core curriculum: Page 30-31

**Geometry Strand**

*Students will use visualization and spatial reasoning to analyze characteristics and properties of geometric shapes.*

**Shapes**

- 2.G.1 Experiment with slides, flips, and turns to compare two-dimensional shapes
- 2.G.2 Identify and appropriately name two-dimensional shapes: circle, square, rectangle, and triangle (both regular and irregular)
- 2.G.3 Compose (put together) and decompose (break apart) two-dimensional shapes

*Students will identify and justify geometric relationships, formally and informally.*

**Geometric Relationships**

- 2.G.4 Group objects by like properties

*Students will apply transformations and symmetry to analyze problem solving situations.*

**Transformational Geometry**

- 2.G.5 Explore and predict the outcome of slides, flips, and turns of two-dimensional shapes
- 2.G.6 Explore line symmetry

1st grade Math core curriculum: Page 23

**Geometry Strand**

*Students will use visualization and spatial reasoning to analyze characteristics and properties of geometric shapes.*

**Shapes**

- 1.G.1 Match shapes and parts of shapes to justify congruency
- 1.G.2 Recognize, name, describe, create, sort, and compare two-dimensional and three-dimensional shapes

*Students will apply transformations and symmetry to analyze problem solving situations.*

**Transformational Geometry**

- 1.G.3 Experiment with slides, flips, and turns of two-dimensional shapes
- 1.G.4 Identify symmetry in two-dimensional shapes

*Students will apply coordinate geometry to analyze problem solving situations.*

**Coordinate Geometry**

- 1.G.5 Recognize geometric shapes and structures in the environment
Appendix B: Quadrilaterals Unit Objectives

Unit 4: Quadrilaterals and Other Polygons

Understand:

(U1) Similarity and congruence guarantees specific relationships among sides and angles.
(U2) Geometric properties define, describe and classify polygons.
(U3) Any polygon can be divided into strategic triangles, and thus properties of triangles can be used for any polygon.

Know:

See Standards plus:

Vocabulary: Polygon vocabulary from triangle unit (make the connections), quadrilaterals, parallelograms, rectangles, rhombuses, squares, trapezoids, medians, pentagon, hexagon, heptagon, octagon, nonagon, decagon, n-gon (all understandings)

For extensions: Kite (U3)
Hierarchy of quadrilaterals (U2)
Area formulas (OU3, U3)
Sum of interior angles formula (OU3, U3)
Properties of specific quadrilaterals (U2)
Coordinate proofs (OU2, OU3, U2)

Do:
See Standards:

G.G.36 Investigate, justify, and apply theorems about the sum of the measures of the interior and exterior angles of polygons (U3)
G.G.37 Investigate, justify, and apply theorems about each interior and exterior angle measure of regular polygons (U3)
G.G.38 Investigate, justify, and apply theorems about parallelograms involving their angles, sides, and diagonals (U2, U3)
G.G.39 Investigate, justify, and apply theorems about special parallelograms (rectangles, rhombuses, squares) involving their angles, sides, and diagonals (U2)
G.G.40 Investigate, justify, and apply theorems about trapezoids (including isosceles trapezoids) involving their angles, sides, medians, and diagonals (U2)
G.G.41 Justify that some quadrilaterals are parallelograms, rhombuses, rectangles, squares, or trapezoids (U2)
G.G.69 Investigate, justify, and apply the properties of triangles and quadrilaterals in the coordinate plane, using the distance, midpoint, and slope formulas (U3)
Appendix C: Daily Objectives of Quadrilaterals Unit

Linked to van Hiele levels

<table>
<thead>
<tr>
<th>Instructional Phases Key:</th>
<th>Skill Types Key:</th>
</tr>
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<tbody>
<tr>
<td>Inf = Information</td>
<td>Vis = Visual</td>
</tr>
<tr>
<td>G.O. = Guided Orientation</td>
<td>Draw = Drawing</td>
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<tr>
<td>Exp = Explicitation</td>
<td>Ver = Verbal</td>
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<tr>
<td>F.O. = Free Orientation</td>
<td>Log = Logical</td>
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<tr>
<td>Int = Integration</td>
<td>App = Applied</td>
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<tr>
<th>Learning Objectives</th>
<th>Inf</th>
<th>G. O.</th>
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<td>Intro to Quadrilaterals:</td>
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<td>• Activate prior knowledge on the 6 types.</td>
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<td>• Use “are” and “are not” examples to refine definitions.</td>
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<td>• Address hierarchy: explain statements such as: &quot;Is a square always a rectangle? Is a rectangle always a square?&quot;</td>
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<td><strong>Computer Lab:</strong> Discover Properties of Parallelograms relating to angles, sides and diagonals</td>
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<td><strong>Homework:</strong> recall and apply these properties in problem solving</td>
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<tr>
<td><strong>Computer Lab:</strong> Discover Properties of Special Parallelograms (rhombus, rectangle and square) relating to angles, sides and diagonals</td>
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<td><strong>Homework:</strong> recall and apply these properties in problem solving</td>
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<td><strong>Classroom:</strong> Discover Properties of Trapezoids relating to angles, sides and diagonals</td>
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<td><strong>Homework:</strong> recall and apply these properties in problem solving</td>
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<td>Consolidate learning through hierarchical graphic organizer of parallelogram family properties</td>
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<td>Day 6</td>
<td>Consolidate learning through matrix organizer of parallelogram family properties. Guess Who – the Quad version: Identify minimum sets of characteristics sufficient to uniquely identify the quadrilateral. Apply to parallelogram proof with coordinate geometry.</td>
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<td>Day 7</td>
<td>Coordinate Geometry Proof on Rectangles: Identify sufficient sets of characteristics that can be quantified with coordinate geometry measures. Emphasize the structure of the logical conclusion</td>
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<td>Day 8</td>
<td>Coordinate Geometry Proof on Rhombi: Identify sufficient sets of characteristics that can be quantified with coordinate geometry measures. Formative assessment on Rectangle proof.</td>
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<td>Day 9</td>
<td>Error Analysis on sample Coordinate Geometry Conclusions. Coordinate Geometry Proof on Squares: Identify sufficient sets of characteristics that can be quantified with coordinate geometry measures.</td>
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<td>Day 10</td>
<td>Parallelogram Family Properties Quiz</td>
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<td>Day 11</td>
<td>Coordinate Geometry Proof on Trapezoids: Identify sufficient sets of characteristics that can be quantified with coordinate geometry measures.</td>
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<td>Day 12  &amp; 13</td>
<td>Mixed Review leading into Winter break</td>
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<td>Day 14</td>
<td>Reactivate prior learning: Create application problem from selected Quad Property</td>
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<td>Day 15</td>
<td>Ways to find the area of regular and irregular geometric objects</td>
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<td>Day 16</td>
<td>Applying quadrilateral properties to find missing dimensions when determining the area of irregular geometric objects</td>
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<td>Day 17</td>
<td>Polygons - definitions &amp; internal, external angle characteristics</td>
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<td>Day 18</td>
<td>Application Assessment: Landscape Project Costing</td>
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<td>Day 19</td>
<td>Explaining why the diagonal properties are true: Using quad definitions in congruent triangle deductive proof, using CPCTC</td>
<td>Lvl 4</td>
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<td>Day 20</td>
<td>Review for unit test</td>
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<td>Day 21</td>
<td>Quadrilaterals Unit Test</td>
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Appendix D: Explaining Quadrilateral Properties

Over the past weeks, we have learned about many different properties of quadrilaterals. We discovered some of these properties using dynamic geometry software examples on the computer. By moving the vertices, we saw that the patterns held for many different examples, but do we couldn’t check every example, so how can we be sure that the pattern is true for ALL examples.

Also, how can we explain why these patterns happen?

Since every polygon can be “cut up” into triangles, we can apply our triangle proof skills to confirm and explain these observations in a way that verifies it for ALL cases.

1. Start from the definition of a parallelogram (both pairs of opposite sides are parallel) to prove that both pairs of opposite sides are congruent.

   Given: ABCD is a parallelogram

   Prove: AB ≅ CD and AD ≅ BC

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ABCD is a parallelogram</td>
<td>1.</td>
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<tr>
<td>2. AB // CD and AD // BC</td>
<td>2.</td>
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<tr>
<td>3. ( \angle ADB \cong \angle CBD ) and ( \angle ABD \cong \angle CDB )</td>
<td>3.</td>
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<tr>
<td>4. DB ≅ ( \angle DB )</td>
<td>4.</td>
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<tr>
<td>5. ( \Delta ABD \cong \Delta CDB )</td>
<td>5.</td>
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<tr>
<td>6. AB ≅ CD and AD ≅ BC</td>
<td>6.</td>
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</table>

Why does this proof show that opposite sides are congruent for all parallelograms?
2. Now you can use the parallelogram definition and the property you proved: “opposite sides are parallel” and “opposite sides are congruent” to prove that the diagonals of a parallelogram bisect each other.

Given: ABCD is a parallelogram

Prove: AE \equiv CE and DE \equiv BE

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
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<tbody>
<tr>
<td>1. ABCD is a parallelogram</td>
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<tr>
<td>2. AB \parallel CD</td>
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<tr>
<td>3. \angle ABE \equiv \angle CDE</td>
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<tr>
<td>4. \angle AEB \equiv \angle CED</td>
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<tr>
<td>5. AB \equiv CD</td>
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<tr>
<td>6. \triangle ABE \equiv \triangle CDE</td>
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<tr>
<td>7. AE \equiv CE and DE \equiv BE</td>
<td>7.</td>
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</tbody>
</table>
3. Use the definition of a rectangle: a parallelogram that is equiangular and any fact about parallelograms that we already proved to prove that the diagonals of a rectangle are congruent.

Given: \(ABCD\) is a rectangle

Prove: \(AC \equiv BD\)

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<th>Statements</th>
<th>Reasons</th>
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<tr>
<td>1. (AD \equiv BC)</td>
<td>1. Given</td>
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<tr>
<td>2. (\angle ADC \equiv \angle BCD)</td>
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<td>3. (\Delta _ _ _ _ \equiv \Delta _ _ _ _)</td>
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<tr>
<td>4. (_ _ _ _ \equiv \Delta _ _ _ _)</td>
<td>4. Reflexive Property</td>
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<td>5. (AC \equiv BD)</td>
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4. Use the definition of a rhombus: a parallelogram that is equilateral and any fact about parallelograms that we already proved to prove that the diagonal of a rhombus bisects each opposite angle.

Given: \(ABCD\) is a rhombus

Prove: \(\angle ADB \equiv \angle CDB\) and \(\angle ABD \equiv \angle CED\)

<table>
<thead>
<tr>
<th>Statements</th>
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<tr>
<td>1. (_ _ _ _ )</td>
<td>1. Given</td>
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<td>2. (_ _ _ _ )</td>
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<td>3. (_ _ _ _ _ \equiv _ _ _ _ )</td>
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<td>4. (_ _ _ _ \equiv _ _ _ _)</td>
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<td>5. (_ _ _ _ )</td>
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5. Use the definition of a rhombus: a parallelogram that is equilateral and any fact about rhombuses that we already proved to **prove that the diagonals of a rhombus are perpendicular**.

Given: ABCD is a rhombus

Prove: AC ⊥ BD

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6. Prove one of these properties, setting up the appropriate Given, Prove and diagram:
   a. The opposite angles of a parallelogram are congruent
   b. The diagonals of an isosceles trapezoid are congruent
   c. Consecutive angles of a parallelogram are supplementary
Appendix E: Computer Lab Sheets and Screen Prints

Geometry
Exploring Properties of Parallelograms

Name: ____________________  Date: ______

Today you will work with dynamic geometry sketches to discover some patterns in the angles, sides and diagonals of parallelograms.

Keep in mind that anything we find to be true for a parallelogram will also be true for a rhombus, rectangle or square since these quadrilaterals all fit the definition of a parallelogram.

Open Internet Explorer and begin at this website:

http://www.keypress.com/x19430.xml

Start with the sketch at the top of the webpage.

Notice that dragging any of the red vertices L, O or V can change the shape of the parallelogram. Try this – drag any vertex – and notice how the angle measures are updated to the right.

1. Which angle is opposite $\angle VEL$?

2. What do you notice about the measures of these two angles?

3. Does this pattern hold true for the other pair of angles?

4. Try to find a counterexample – a parallelogram that breaks this pattern – by moving around the vertices. What types of things did you try? (sketch at least 2)

Could you create a parallelogram for which this pattern did not hold?

5. Summarize the pattern you have observed with words and a diagram:

| In any parallelogram, opposite angles are _________ |
6. Remember that \textit{consecutive} angles are angles that are \textit{one after the other}, so they \textit{share a common side}. Click on “Show Sums”. What is special about these pairings of angles?

7. Change the shape of the parallelogram. Can you find a parallelogram that changes the sum of the angle pairs?

8. Summarize the pattern you have observed with words and a diagram:

In any parallelogram,
consecutive angles are ____________.

Now scroll down the webpage to the 2\textsuperscript{nd} sketch. Do not click “Show Diagonals” yet. (If you did by accident, reload the page to reset it)

Notice that this sketch shows the lengths of the sides in terms of pixels (the size of a “dot” on the computer screen)

9. Change the shape of the parallelogram and watch the lengths of the opposite sides. What do you notice about sides LO and VE? OV and EL?

The opposite sides of a parallelogram are __________.
10. Now click on “Show Diagonals”.  
What do segments LM and VM represent?

What do segments OM and EM represent?

11. Drag the red vertices to change the shape of the parallelogram. Explain what you notice about these diagonal segments.

12. Capture your observations in words and a diagram:

Point M is the __________ of diagonal LV  
and M is also the __________ of diagonal OE.

Diagonal LV __________ diagonal OE,  
And diagonal OE __________ diagonal LV.

Or, stated in fewer, more general words:

The diagonals of a parallelogram ____________________:

13. Yesterday we reviewed that a rhombus, rectangle or square each fit the definition of a parallelogram. Given that, do these properties of parallelograms also hold true for rhombuses, rectangles and squares? Explain your reasoning.
**Sketch**

This sketch shows parallelogram $LOVE$ and the measures of its angles. Drag vertices $L$, $O$, or $V$ to change the parallelogram. If you drag a point off the screen, press Start Over to return it.

$m(\text{Angle LOV}) = 139.274^\circ$
$m(\text{Angle OVE}) = 40.726^\circ$
$m(\text{Angle VEL}) = 139.274^\circ$
$m(\text{Angle ELO}) = 40.726^\circ$

**Sketch**

This sketch shows parallelogram $LOVE$ and the lengths of its sides. Drag vertices $L$, $O$, or $V$ to change the parallelogram. If you drag a point off the screen, press Start Over to return it.

$LO = 82.638$ pixels
$OV = 108.167$ pixels
$VE = 82.638$ pixels
$EL = 100.167$ pixels
Today you will work with dynamic geometry sketches to discover some patterns that are unique to rhombuses, rectangles and squares.

Open Internet Explorer and begin at this website: http://www.keypress.com/x19432.xml

Start with the 2nd sketch on the webpage – of rhombus RHMB.
Don’t click on “Show Angles at Vertices” yet – if you do, reload the webpage.

1. Recall the definition of a rhombus:
   a parallelogram that has

2. Only the measure of angle RXH is given. What is it?
   Drag vertices R, H and M to change the shape of the rhombus. Does this change the measure of angle RXH?

3. What are the measures of the other angles formed by the intersection of the two diagonals:
   \( m \angle HXM = \) \_
   \( m \angle MXB = \) \_
   \( m \angle BXR = \) \_
   Based on these angle measures, what can we say about the diagonals?

4. You figured out yesterday that the diagonals of a parallelogram bisect each other. Do the diagonals of this rhombus bisect each other? Why?

The diagonals of a rhombus have two special characteristics:

1) They \__________________\.

2) They \__________________\.
The diagonals and the sides of the rhombus form two angles at each vertex. Click on "Show Angles At Vertices" to see the measures of these angles.

5. How does each pair of angles compare? Change the size and shape of the rhombus to see whether this is always true.

6. Summarize your observations in words and a diagram:

The diagonals _________ the angles of the rhombus.
Now move down to the last Sketch on the webpage, of rectangle RECT.

7. Recall the definition of a rectangle:
   a parallelogram that has ______________________

8. Compare the lengths of diagonals RC and ET. What do you notice? Change the rectangle — is this always true?

9. You figured out yesterday that the diagonals of a parallelogram bisect each other. Do the diagonals of this rectangle bisect each other? Why?

The diagonals of a rectangle have two special characteristics:

1) They ________________________.
2) They ________________________.

[Diagram of a rectangle with diagonal lines forming triangles]
A square is a parallelogram, as well as both a rectangle and a rhombus. Therefore, the square inherits all the properties of each of these quadrilaterals.

The diagonals of a *square* have three special characteristics:

1) They ____________________.
2) They ____________________.
3) They ____________________.

10. Go back up to the 2nd *sketch* of rhombus RHMB. Drag the vertices to make it into a square.
   What is the measure of each angle formed by the diagonal and a side?
   Like you found for the rhombus,

   The diagonals _______ the angles of the *square*.
**Sketch**

This sketch shows parallelogram LOVE formed by two, intersecting pairs of parallel lines. Both pairs of parallel lines are the same distance apart. Drag vertex E to change the position and angle of intersection of the lines, and use the red segment to change the distance between the pairs of parallel lines. If you drag a point off the screen, press Start Over to return it.

- LO = 79.3 units
- OV = 79.3 units
- VE = 79.3 units
- EL = 79.3 units

**Sketch**

This sketch shows rhombus RHMB and its diagonals, which intersect at point X. Drag vertices R, H, or M to change the size and shape of the rhombus. If you drag a point off the screen, press Start Over to return it.

- $\text{mAngle RXH} = 90.0^\circ$
Sketch
This sketch shows rectangle RECT and its diagonals. Drag vertices R, E, or C to change the size and shape of the rectangle. Press Start Over if you drag a vertex too far.

RC = 5.66 units
ET = 5.66 units