11-1998

Solitary waves for two and three coupled nonlinear Schrödinger equations

Foek T. Hioe
Saint John Fisher College, fhioe@sjfc.edu

Publication Information

Hioe, Foek T. (1998). "Solitary waves for two and three coupled nonlinear Schrödinger equations." *Physical Review E* 58.5, 6700-6707.

Solitary waves for two and three coupled nonlinear Schrödinger equations

F. T. Hioe
Department of Physics, St. John Fisher College, Rochester, New York 14618
(Received 27 March 1998)

We present solitary-wave solutions of two and three coupled nonlinear Schrödinger equations when the waves propagate in the normal and anomalous group-velocity dispersion regions. A wave of the form $\mathrm{sech}^2\,\xi-\frac{2}{3}$ is found, which, together with two known waves of the forms $\tanh\xi\,\mathrm{sech}\,\xi$ and $\mathrm{sech}^2\,\xi$, are shown to form a new generation of complementary waves. The implication of this wave set and its applications to coupled solitary-wave propagation is discussed. [S1063-651X(98)07411-X]

PACS number(s): 42.65.Tg, 42.81.Dp, 03.40.Kf

I. INTRODUCTION

Because of its many useful applications in nonlinear optics [1], the problem of two interacting optical waves that satisfy two coupled nonlinear Schrödinger (NLS) equations and that are shape-preserving (solitary) has been extensively studied for many years. The well-known cases of interaction between two solitary waves are (i) two bright solitary waves that propagate in the anomalous group-velocity dispersion (GVD) region [2], (ii) one bright solitary wave in the anomalous GVD region that interacts with a dark solitary wave in the normal GVD region [3], (iii) one bright solitary wave in the normal GVD region that interacts with a dark solitary wave in the anomalous GVD region [4], and (iv) two dark solitary waves that propagate in the normal GVD region [5]. The simplest bright and dark solitary waves have the forms $\mathrm{sech}(\alpha\xi)$ and $\tanh(\alpha\xi)$, respectively, where α is some constant and $\xi=t-z/v$, with t, z, and v denoting the time, displacement, and velocity. A solitary wave pair that consists of a product type of the forms $\tanh(\alpha\xi)\mathrm{sech}(\alpha\xi)$ and $\mathrm{sech}^2(\alpha\xi)$ was given by Tratnik and Sipe [6], that of the forms $\tanh(\alpha\xi)\mathrm{sech}^{s-1}(\alpha\xi)$ and $\mathrm{sech}^s(\alpha\xi)$, where s is between 1 and 2, was given by Silberberg and Barad [7], and that where both waves are asymmetric but which reduce under certain limits to the forms $\tanh(\alpha_1\xi)\mathrm{sech}(\alpha_2\xi)$ and $\mathrm{sech}(\alpha_1\xi)$ was given by Christodoulides and Joseph [8]. Solitary waves, each of which is a superposition of bright and dark solitary waves, were given by the author [9], and many periodic solitary waves, which are expressed in terms of Jacobian elliptic functions or their products, were given by several authors [9–12].

Let us denote the simplest forms of dark and bright solitary waves by

$$f_1(\xi)=\tanh(\alpha\xi)$$

and

$$f_2(\xi)=\mathrm{sech}(\alpha\xi), \tag{1}$$

respectively. We refer to Eq. (1) as the first generation of solitary wave set.

In this paper, we present a number of coupled solitary waves for two and three coupled NLS equations that propagate in the normal and in the anomalous GVD regions. First we present a solitary wave pair for two coupled waves that propagate in the normal GVD region. The pair consists of a solitary wave of the form $\mathrm{sech}^2(\alpha\xi)-\frac{2}{3}$. Indeed, we shall show that the set of three solitary waves of the forms

$$g_1(\xi)=\mathrm{sech}^2(\alpha\xi)-\tfrac{2}{3},$$

$$g_2(\xi)=\tanh(\alpha\xi)\mathrm{sech}(\alpha\xi),$$

$$g_3(\xi)=\mathrm{sech}^2(\alpha\xi) \tag{2}$$

can be appropriately considered as the second generation of solitary wave set. This generation consists of the three members of Eq. (2) (which we shall call red, white, and blue solitary waves for easy reference, as opposed to dark and bright solitary waves of the first generation), and we show that it is one of the simplest sets of ''complementary'' solitary-wave solutions, i.e., a solution that consists of three *different* wave forms, for three coupled NLS equations. Subsets of it appear as solutions for two coupled NLS equations: the white-blue or (g_2,g_3) combination given by Tratnik and Sipe [6], and the red-white or (g_1,g_2) combination that we present in this paper. The significance of this wave form and the realization that it is one of the solitary waves for a solution of three coupled NLS equations will be further amplified in the following sections.

II. *N* COUPLED NONLINEAR SCHRÖDINGER EQUATIONS

When two optical waves of different frequencies copropagate in a medium and interact nonlinearly through the medium, the propagation equation for slowly varying complex amplitude $\phi_m(z,t)$ of the mth electric field is [1]

$$i\phi_{mz}+i\beta_{1m}\phi_{mt}-\frac{\beta_{2m}}{2}\,\phi_{mtt}+\frac{i\alpha_m}{2}\,\phi_m$$
$$+\gamma_m(f_{mm}|\phi_m|^2+2\,f_{mm'}|\phi_{m'}|^2)\phi_m=0,$$
$$m=1,2,\quad m'\neq m, \tag{3}$$

where $\beta_{1m}=1/v_{gm}$, v_{gm} is the group velocity, β_{2m} is the group-velocity dispersion (GVD) coefficient, α_m is the loss coefficient, γ_m is the nonlinear coefficient, and $f_{mm'}$ is the overlap integral, and where the subscripts in z and t denote derivatives with respect to z and t as opposed to the subscript

m for different components. The medium is said to exhibit normal dispersion if $\beta_2>0$, anomalous dispersion if $\beta_2<0$.

If the nonlinear coupling is between two polarization components $\phi_m(z,t)$, $m=1,2$, of a wave at some central frequency, the propagation equations are

$$\begin{aligned} &i\phi_{1z}+i\beta_{11}\phi_{1t}-\frac{\beta_2}{2}\phi_{1tt}+\frac{i\alpha}{2}\phi_1+\gamma[(|\phi_1|^2+p|\phi_2|^2)\\ &\quad\times\phi_1+q\phi_1^*\phi_2^2e^{-2i\Delta\beta z}]=0,\\ &i\phi_{2z}+i\beta_{12}\phi_{2t}-\frac{\beta_2}{2}\phi_{2tt}+\frac{i\alpha}{2}\phi_2+\gamma[(|\phi_2|^2+p|\phi_1|^2)\\ &\quad\times\phi_2+q\phi_2^*\phi_1^2e^{2i\Delta\beta z}]=0, \end{aligned}\tag{4}$$

where $\Delta\beta=\beta_{11}-\beta_{12}$ is the wave-vector mismatch due to, for example, the birefringence of the medium through which the wave propagates, and the parameters p and q satisfy $p+q=1$. For a medium such as an optical fiber with a relatively large birefringence, the wave propagation equations can be approximated by

$$\begin{aligned} &i\phi_{1z}+i\beta_{11}\phi_{1t}-\frac{\beta_2}{2}\phi_{1tt}+\frac{i\alpha}{2}\phi_1+\gamma(|\phi_1|^2+B|\phi_2|^2)\\ &\quad\times\phi_1=0,\\ &i\phi_{2z}+i\beta_{12}\phi_{2t}-\frac{\beta_2}{2}\phi_{2tt}+\frac{i\alpha}{2}\phi_2+\gamma(|\phi_2|^2+B|\phi_1|^2)\\ &\quad\times\phi_2=0, \end{aligned}$$

where B can vary between 2/3 and 2, and these equations are similar to Eqs. (3).

If the two coupled waves or components propagate with approximately the same group velocity v, the $i\beta_{1m}\phi_{mt}$ terms in Eqs. (3) and (4) can be eliminated by the transformation $t\to t-z/v$, and Eqs. (3) and (4) become coupled nonlinear Schrödinger-like equations.

The two sets of equations (3) and (4) are mathematically similar, and their generalization to $N(>2)$ coupled waves or components can be written down. Analytic solutions, mainly in the form of coupled solitary waves, are possible only for some special cases. The analytic solitary waves we shall present in this paper for $N=2$ and 3 are also only applicable to some special cases. However, they could provide some useful guides for studies of two or three nonlinearly coupled waves that propagate under conditions that are not too different from the physical conditions that have been assumed to permit these solitary waves.

Instead of writing down the general N coupled wave equations, we begin with the following two equivalent sets, Eqs. (5) and (6) below, of N coupled wave equations, which can be seen to reduce, for $N=2$, to Eqs. (3) (with specific values for f's) and (4), respectively. Consider N coupled equations for the slowly varying complex amplitudes or components $\phi_m(z,t)$, $m=1,2,\ldots,N$ of the electric fields propagating along the z axis that satisfy the following coupled nonlinear Schrödinger-like equations:

$$\begin{aligned} &i\phi_{mz}+\phi_{mtt}+\kappa_m\phi_m+p_m\left(\sum_{j=1}^{N}|\phi_j|^2\right)\phi_m+q_m\left(\sum_{j=1}^{N}\phi_j^2\right)\\ &\quad\times\phi_m^*=0,\quad m=1,\ldots,N, \end{aligned}\tag{5}$$

where p's and q's are dimensionless parameters that we assume satisfy the relation $p_m+q_m=\varepsilon$, $\varepsilon=+1$ or -1. A closely related set of coupled equations is

$$\begin{aligned} &i\psi_{mz}+\psi_{mtt}+p_m\left(\sum_{j=1}^{N}|\psi_j|^2\right)\psi_m\\ &\quad+q_m\left(\sum_{j=1}^{N}\psi_j^2e^{2i\kappa_j z}\right)\psi_m^*e^{-2i\kappa_m z}=0,\quad m=1,\ldots,N \end{aligned}\tag{6}$$

which can be transformed into Eq. (5) with the substitutions $\psi_m=\phi_m\exp(-i\kappa_m z)$. We first search for the stationary-wave solution of the form

$$\phi_m(z,t)=x_m(t)\exp(i\Omega z),\tag{7}$$

where Ω is a real constant, and $x_m(t)$ are real functions of t only. Equations (5) reduce to the following, which we call the associated dynamical coupled nonlinear Schrödinger equations:

$$\ddot{x}_m-A_mx_m+\varepsilon\left(\sum_{j=1}^{N}x_j^2\right)x_m=0,\quad m=1,\ldots,N,\tag{8}$$

where $A_m=\Omega-\kappa_m$, $\dot{x}$ denotes dx/dt, and $\varepsilon=+1$ or -1. Because Eqs. (5) and (6) are invariant under a Galilean transformation, traveling waves can be constructed from Eq. (7) by replacing $\phi_m(z,t)$ by

$$\phi_m(z,t-z/v)\exp[i(t-z/2v)/(2v)],\tag{9}$$

where v is the velocity of the waves.

The case $\varepsilon=+1$, $N=1$ for Eq. (5) [or Eq. (6)] can be identified with the standard NLS equation for waves that propagate in the anomalous GVD region and one that gives the bright solitary wave; and the case $\varepsilon=-1$, $N=1$ can be seen to be equivalent to the standard equation for waves that propagate in the normal GVD region and one that gives the dark solitary wave. For $N>1$, it should be noted that Eqs. (5) have either $\varepsilon=+1$ or $\varepsilon=-1$ for *all* N equations, i.e., where all N coupled waves propagate either in the anomalous ($\varepsilon=+1$) GVD region or in the normal ($\varepsilon=-1$) GVD region, not any ''mixed'' cases where one or more of the equations have $\varepsilon=+1$ and -1. To eliminate the permutation symmetry, we assume that the equations in (8) have been arranged according to $A_1\leqslant A_2\leqslant\cdots\leqslant A_N$.

III. TWO COUPLED NLS EQUATIONS

We first present nine periodic (or elliptic) solutions in terms of Jacobian elliptic functions of modulus k for $N=2$, $\varepsilon=+1$ of Eqs. (8). Seven of these that do not include those with the same wave forms were given earlier [12], but some of them were not expressed in the most simplified form in Ref. [12]. We present the complete set below, which we number (I) to (IX); note in particular solutions (III) and (V) and solutions (VIII) and (IX), which are expressed in simpler forms more suitable for comparisons with our other results later. The modulus k of the elliptic functions given below is in the range $0<k^2\leqslant 1$ unless otherwise specified.

Solution (I):

$$x_1=C_1\mathrm{sn}(\alpha t,k),\quad x_2=C_2\mathrm{cn}(\alpha t,k),$$

$$\alpha^2=(A_2-A_1)/k^2,\quad C_1^2=A_2+\alpha^2-2\alpha^2k^2,$$

$$C_2^2=A_2+\alpha^2,\quad A_2>A_1.$$

Solution (II):

$$x_1=C_1k\,\mathrm{sn}(\alpha t,k),\quad x_2=C_2\mathrm{dn}(\alpha t,k),$$

$$\alpha^2=A_2-A_1,\quad C_1^2=A_2+\alpha^2k^2-2\alpha^2,$$

$$C_2^2=-A_2-\alpha^2k^2,\quad A_2>A_1.$$

Solution (III):

$$x_1=C_1\mathrm{cn}(\alpha t,k),\quad x_2=C_2\mathrm{cn}(\alpha t,k),$$

$$\alpha^2=A_1/(2k^2-1),\quad C_1^2+C_2{}^2=2\alpha^2k^2,\quad A_1=A_2,$$

$$A_1>0\ \text{ for }\ k^2>1/2,\ A_1<0\ \text{ for }\ k^2<1/2.$$

Solution (IV):

$$x_1=C_1(k/k')\mathrm{cn}(\alpha t,k),\quad x_2=C_2(1/k')\mathrm{dn}(\alpha t,k),$$

$$\alpha^2=(A_2-A_1)/k'^2,\quad C_1^2=-A_2-\alpha^2k^2+2\alpha^2,$$

$$C_2^2=A_2-\alpha^2k^2,\quad A_2>A_1.$$

Solution (V):

$$x_1=C_1\mathrm{dn}(\alpha t,k),\quad x_2=C_2\mathrm{dn}(\alpha t,k),$$

$$\alpha^2=A_1/(2-k^2),\quad C_1^2+C_2^2=2\alpha^2,\quad A_1=A_2>0.$$

Solution (VI):

$$x_1=Ck\,\mathrm{sn}(\alpha t,k)\mathrm{cn}(\alpha t,k),\quad x_2=C\,\mathrm{cn}(\alpha t,k)\mathrm{dn}(\alpha t,k),$$

$$\alpha^2=(A_2-A_1)/3,\quad k^2=(4A_2-A_1)/[5(A_2-A_1)],$$

$$C^2=2(4A_2-A_1)/5,\quad A_2\geqslant 4A_1.$$

Solution (VII):

$$x_1=C\,\mathrm{sn}(\alpha t,k)\mathrm{dn}(\alpha t,k),\quad x_2=C\,\mathrm{cn}(\alpha t,k)\mathrm{dn}(\alpha t,k),$$

$$\alpha^2=(4A_2-A_1)/15,\quad k^2=5(A_2-A_1)/[(4A_2-A_1)],$$

$$C^2=2(4A_2-A_1)/5,\quad A_1<A_2\leqslant 4A_1.$$

Solution (VIII):

$$x_1=Ck^2\mathrm{sn}(\alpha t,k)\mathrm{cn}(\alpha t,k),\quad x_2=C[\tfrac{1}{3}G_+-k^2\mathrm{sn}^2\alpha t,k)],$$

TABLE I. Solutions of Eqs. (8) for $N=2$, $\varepsilon=+1$.

	(1)	(2)	(3)
x_1	C_1f_1	C_1f_2	C_1g_2
x_2	C_2f_2	C_2f_2	C_2g_3
A's	$A_2>A_1>0$	$A_1=A_2>0$	$A_2=4A_1>0$
α^2	A_2-A_1	A_1	A_1
C's	$C_1^2=A_1$	$C_1^2+C_2^2=2A_1$	$C_1^2=C_2^2=6A_1$
	$C_2^2=2A_2-A_1$		

$$\alpha^2=\frac{1}{10}\left[\sqrt{\tfrac{5}{3}(A_2^2-A_1^2)}+2A_2-3A_1\right]$$

$$k^2=\frac{2\sqrt{\tfrac{5}{3}(A_2^2-A_1^2)}}{\sqrt{\tfrac{5}{3}(A_2^2-A_1^2)}+2A_2-3A_1},$$

$$C=\frac{3}{5}\,\frac{\sqrt{\tfrac{5}{3}(A_2^2-A_1^2)}+2A_2-3A_1}{\sqrt{2(A_2-A_1)}},$$

$$\tfrac{1}{3}G_+=\left[\frac{3}{2}-\frac{1}{2}\left(\frac{3(A_2+A_1)}{5(A_2-A_1)}\right)^{1/2}\right]^{-1},$$

$$A_2\geqslant 4A_1.$$

Solution (IX):

$$x_1=Ck\,\mathrm{sn}(\alpha t,k)\mathrm{dn}(\alpha t,k),\quad x_2=C[\tfrac{1}{3}G_+-k^2\mathrm{sn}^2(\alpha t,k)],$$

$$\alpha^2=\sqrt{\tfrac{1}{15}(A_2^2-A_1^2)},\quad k^2=\frac{\sqrt{\tfrac{5}{3}(A_2^2-A_1^2)}+2A_2-3A_1}{2\sqrt{\tfrac{5}{3}(A_2^2-A_1^2)}},$$

$$C=\sqrt{\tfrac{6}{5}(A_1+A_2)},\quad \tfrac{1}{3}G_+=\frac{1}{2}+\frac{1}{2}\left(\frac{5(A_2-A_1)}{3(A_2+A_1)}\right)^{1/2},$$

$$8A_1/7<A_2\leqslant 4A_1.$$

It should be noted that whenever the two solitary waves are of the same form, it necessarily requires that the corresponding A's in Eqs. (8) must be equal. This is one reason that the use of different or complementary wave forms is sometimes advantageous or necessary as it permits more freedom in the choice of parameters compared to the use of the same wave form. These nine solutions reduce to only three distinct solutions in terms of (1) and (2) when $k^2=1$, and they are given in Table I. Solution (3) of Table I, which gives the solitary wave pair (g_2,g_3) involving a subset of the second generation of solitary wave set was first given by Tratnik and Sipe [6].

Next, we present below five elliptic solutions for $N=2$, $\varepsilon=-1$, of Eqs. (8), which we number (i)–(v). Both A_1 and A_2 are assumed to be <0.

TABLE II. Solutions of Eqs. (8) for $N=2$, $\varepsilon=-1$.

	(1)	(2)	(3)
x_1	C_1f_1	C_1f_1	C_1g_1
x_2	C_2f_1	C_2f_2	C_2g_2
A's	$A_1=A_2<0$	$A_2<0,\ A_2>A_1$	$A_1=\frac{8}{7}A_2<0$
α^2	$-A_1/2$	A_2-A_1	$-A_1/8$
C's	$C_1^2+C_2^2=-A_1$	$C_1^2=-A_1$ $C_2^2=-2A_2+A_1$	$C_1^2=C_2^2=-9A_1/4$

Solution (i):

$$x_1=C_1\mathrm{sn}(\alpha t,k),\quad x_2=C_2\mathrm{sn}(\alpha t,k),$$

$$\alpha^2=-A_1/(1+k^2),\quad C_1^2+C_2^2=2\alpha^2k^2,\quad A_1=A_2.$$

Solution (ii):

$$x_1=C_1\mathrm{sn}(\alpha t,k),\quad x_2=C_2\mathrm{cn}(\alpha t,k),$$

$$\alpha^2=(A_2-A_1)/k^2,\quad C_1^2=-A_2-\alpha^2+2\alpha^2k^2,$$

$$C_2^2=-A_2-\alpha^2,\quad A_2>A_1.$$

Solution (iii):

$$x_1=C_1k\,\mathrm{sn}(\alpha t,k),\quad x_2=C_2\mathrm{dn}(\alpha t,k),$$

$$\alpha^2=A_2-A_1,\quad C_1^2=-A_2-\alpha^2k^2+2\alpha^2,$$

$$C_2^2=-A_2-\alpha^2k^2,\quad A_2>A_1.$$

Solution (iv):

$$x_1=C[\tfrac{1}{3}G_--k^2\mathrm{sn}^2(\alpha t,k)],\quad x_2=Ck^2\mathrm{sn}(\alpha t,k)\mathrm{cn}(\alpha t,k),$$

$$\alpha^2=\tfrac{1}{10}\left(\sqrt{\tfrac{5}{3}(A_1^2-A_2^2)}+2A_1-3A_2\right),$$

$$k^2=\frac{2\sqrt{\frac{5}{3}(A_1^2-A_2^2)}}{\sqrt{\frac{5}{3}(A_1^2-A_2^2)}+2A_1-3A_2},$$

$$C=\frac{3}{5}\,\frac{\sqrt{\frac{5}{3}(A_1^2-A_2^2)}+2A_1-3A_2}{\sqrt{2(A_2-A_1)}},$$

$$\frac{1}{3}\,G_-=\left[\frac{3}{2}+\frac{1}{2}\left(\frac{3(A_1+A_2)}{5(A_1-A_2)}\right)^{1/2}\right]^{-1},$$

$$(8/7)A_2\leqslant A_1<A_2.$$

Solution (v):

$$x_1=C[\tfrac{1}{3}G_--k^2\mathrm{sn}^2(\alpha t,k)],\quad x_2=Ck\,\mathrm{sn}(\alpha t,k)\mathrm{dn}(\alpha t,k),$$

$$\alpha^2=\sqrt{\tfrac{1}{15}(A_1^2-A_2^2)},\quad k^2=\frac{\sqrt{\frac{5}{3}(A_1^2-A_2^2)}+2A_1-3A_2}{2\sqrt{\frac{5}{3}\,(A_1^2-A_2^2)}},$$

$$C=\sqrt{-\tfrac{6}{5}(A_1+A_2)},\quad \frac{1}{3}\,G_-=\frac{1}{2}-\frac{1}{2}\left(\frac{5(A_1-A_2)}{3(A_1+A_2)}\right)^{1/2},$$

$$4A_2<A_1\leqslant(8/7)A_2.$$

These five solutions reduce to three distinct solutions in terms of Eqs. (1) and (2) when $k^2=1$, and they are given in Table II. In particular, solution (3) of Table II gives a solitary wave pair (g_1,g_2) that involves a wave form g_1, and that, together with solution (3) of Table I suggests that (g_1,g_2,g_3) may be a triplet of complementary waves that should be considered together. The periodic form of this wave (g_1) is $\frac{1}{3}G_--k^2\mathrm{sn}^2(\alpha t,k)$. Let us first present the solutions of Eqs. (8) for $N=3$ in the next section.

TABLE III. Solutions of Eqs. (8) in f's of Eq. (1) for $N=3$, $\varepsilon=+1$.

	(1)	(2)	(3)
x_1	C_1f_1	C_1f_1	C_1f_2
x_2	C_2f_1	C_2f_2	C_2f_2
x_3	C_3f_2	C_3f_2	C_3f_2
A's	$A_3>A_1=A_2>0$	$A_2=A_3>A_1>0$	$A_1=A_2=A_3>0$
α^2	A_3-A_1	A_2-A_1	A_1
C's	$C_1^2+C_2^2=A_1$ $C_3^2=2A_3-A_1$	$C_1^2=A_1$ $C_2^2+C_3^2=2A_2-A_1$	$C_1^2+C_2^2+C_3^2=2A_1$

IV. SOLUTIONS OF COUPLED NLS EQUATIONS FOR $N=3$

In terms of Eqs. (1) and (2), the aperiodic solutions of Eqs. (8) for $N=3$ are given in Tables III and IV for $\varepsilon=+1$, and in Tables V and VI for $\varepsilon=-1$. In particular, the

TABLE IV. Solutions of Eqs. (8) in g's of Eq. (2) for $N=3$, $\varepsilon=+1$.

	(4)	(5)	(6)
x_1	C_1g_1	C_1g_2	C_1g_2
x_2	C_2g_2	C_2g_2	C_2g_3
x_3	C_3g_3	C_3g_3	C_3g_3
A's	$A_3=4A_2-3A_1,\ A_2>A_1>0$	$A_3=4A_1=4A_2>0$	$A_3=A_2=4A_1>0$
α^2	A_2-A_1	A_1	A_1
C's	$C_1^2=9A_1/4$ $C_2^2=3(2A_2-A_1)$ $C_3^2=3(8A_2-7A_1)/4$	$C_1^2+C_2^2=C_3^2=6A_1$	$C_1^2=C_2^2+C_3^2=6A_1$

TABLE V. Solutions of Eqs. (8) in f's of Eq. (1) for $N=3$, $\varepsilon=-1$.

	(1)	(2)	(3)
x_1	C_1f_1	C_1f_1	C_1f_1
x_2	C_2f_1	C_2f_1	C_2f_2
x_3	C_3f_1	C_3f_2	C_3f_2
A's	$A_1=A_2=A_3<0$	$A_1=A_2<0$	$A_2=A_3<0$
		$A_3>A_1>2A_3$	$A_2>A_1>2A_2$
α^2	$-A_1/2$	A_3-A_1	A_2-A_1
C's	$C_1^2+C_2^2+C_3^2=-A_1$	$C_1^2+C_2^2=-A_1$	$C_1^2=-A_1$
		$C_3^2=-2A_3+A_1$	$C_2^2+C_3^2=-2A_2+A_1$

only complementary solutions, i.e., solutions that consist of *different* wave forms, are solution (4) of Table IV for $\varepsilon=+1$ and solution (6) of Table VI for $\varepsilon=-1$. These two solutions can be conveniently expressed together as

$$x_1=C_1g_1,\quad x_2=C_2g_2,\quad x_3=C_3g_3, \tag{10}$$

where

$$C_1^2=9\varepsilon A_1/4,\quad C_2^2=3\varepsilon(2A_2-A_1),$$

$$C_3^2=3\varepsilon(8A_2-7A_1)/4,\quad \alpha^2=A_2-A_1,$$

$$A_3=4A_2-3A_1,$$

$$A_2>A_1>0 \text{ for } \varepsilon=+1,$$

$$A_1<0,\quad A_2>A_1\geqslant 8A_2/7 \text{ for } \varepsilon=-1.$$

We thus come to the important realization that (g_1,g_2,g_3) is a solution for $N=3$, $\varepsilon=+1$ or -1. It means that even though the red solitary wave (g_1) cannot propagate in the anomalous GVD region with another coupled wave (for $N=2$), and that the blue solitary wave (g_3) cannot propagate in the normal GVD region with another coupled wave (for $N=2$), either of them *can* propagate in the normal or the anomalous GVD region if it is coupled with *two* other waves of different colors.

We may recall a similar situation when we go from $N=1$ to $N=2$ for Eq. (5), where the bright solitary wave is a solution for $N=1$, $\varepsilon=+1$ and not $\varepsilon=-1$, and that the dark solitary wave is a solution for $N=1$, $\varepsilon=-1$ and not $\varepsilon=+1$, but where the coupled bright and dark solitary wave pair can propagate in either the normal or anomalous GVD region, i.e., the bright-dark solitary wave pair is a solution for $N=2$, for $\varepsilon=+1$ or -1. Thus to experimentally realize our solitary wave pair (g_1,g_2) that consists of a wave form g_1, the pair needs to propagate in the normal GVD region, but it can also propagate in the anomalous GVD region if the pair is coupled to g_3. Similarly, the solitary wave pair (g_2,g_3) found by Tratnik and Sipe [6] that can propagate in the anomalous GVD region can be made to propagate in the normal GVD region if the pair is coupled to g_1.

Tables IV and VI show that for $N=3$ two other combinations involving (g_1,g_2) are possible for $\varepsilon=-1$: (g_1,g_2,g_2) and (g_1,g_1,g_2); and two other combinations involving (g_2,g_3) are possible for $\varepsilon=+1$: (g_2,g_2,g_3) and (g_2,g_3,g_3). That means that to send solitary waves of the second generation through a medium, the red solitary wave (g_1) is always needed as one of the coupled waves if the waves are to travel in the normal GVD region, the blue solitary wave (g_3) is always needed as one of the coupled waves if the waves are to travel in the anomalous GVD region, and the white solitary wave (g_2) is always needed as one of the coupled waves in either region.

As in the case for $N=2$, periodic or elliptic solutions can be found for the case $N=3$. We present three such solutions here that reduce to Eq. (10) when $k^2=1$.

Solution (I):

$$x_1=C_1\alpha[\tfrac{1}{3}G_- -k^2\mathrm{sn}^2(\alpha t,k)],$$

$$x_2=C_2\alpha k\,\mathrm{sn}(\alpha t)\mathrm{cn}(\alpha t,k)],$$

$$x_3=C_3\alpha k\,\mathrm{cn}(\alpha t)\mathrm{dn}(\alpha t,k)],$$

where

TABLE VI. Solutions of Eqs. (8) in g's of Eq. (2) for $N=3$, $\varepsilon=-1$.

	(4)	(5)	(6)
x_1	C_1g_1	C_1g_1	C_1g_1
x_2	C_2g_1	C_2g_2	C_2g_2
x_3	C_3g_2	C_3g_2	C_3g_3
A's	$A_1=A_2=8A_3/7<0$	$A_2=A_3=7A_1/8<0$	$A_3=4A_2-3A_1$, $A_1<0$
			$A_2>A_1\geqslant 8A_2/7$
α^2	$-A_1/8$	$-A_1/8$	A_2-A_1
C's	$C_1^2+C_2^2=C_3^2=-9A_1/4$	$C_1^2=C_2^2+C_3^2=-9A_1/4$	$C_1^2=-9A_1/4$
			$C_2^2=3(-2A_2+A_1)$
			$C_3^2=3(-8A_2+7A_1)/4$

$$G_{\pm}=1+k^2\pm(1-k^2+k^4)^{1/2},$$

$$\alpha^2=\tfrac{1}{3}(A_3-A_2),\quad k^2=\tfrac{1}{3}[-(\gamma-2)+2(\gamma^2-\gamma-2)^{1/2}],$$

$$\gamma=(2A_3+A_2-3A_1)/(A_3-A_2),$$

$$C_1^2=\varepsilon[(4A_3-A_2)/(A_3-A_2)-5k^2]/[\tfrac{1}{9}G_-^2-\tfrac{2}{3}G_-k^2+k^4],$$

$$C_3^2=(k^2-\tfrac{2}{3}G_-k^2)C_1^2+6\varepsilon,$$

$$C_2^2=k^2(C_1^2+C_3^2).$$

This solution is applicable in the region $2<\gamma\leqslant 3$, $0<k^2\leqslant 1$, or $A_3\geqslant 4A_2-3A_1$, and $A_2>A_1$. For $\varepsilon=+1$, $A_1>0$, and for $\varepsilon=-1$, $A_2,A_1<0$; and it becomes Eq. (10) when $A_3=4A_2-3A_1$ for which $\gamma=3$ and $k^2=1$. Compared to the aperiodic solutions, periodic solitary-wave solutions permit an additional freedom of choice that can be used to affect the shapes and amplitudes of the waves.

Solution (II):

$$x_1=C_1\alpha[\tfrac{1}{3}G_--k^2\mathrm{sn}^2(\alpha t,k)],$$

$$x_2=C_2\alpha k\,\mathrm{sn}(\alpha t)\mathrm{cn}(\alpha t,k),$$

$$x_3=C_3\alpha[\tfrac{1}{3}G_+-k^2\mathrm{sn}^2(\alpha t,k)],$$

where

$$G_{\pm}=1+k^2\pm(1-k^2+k^4)^{1/2},$$

$$\alpha^2=(A_3-A_2)/(4+k^2-2G_-),$$

$$k^2=(2\gamma^2-2)/[\gamma^2-1+(3\gamma^2-3)^{1/2}],$$

$$\gamma=(A_3-A_1)/(A_3-2A_2+A_1),$$

$$C_1^2=\frac{\varepsilon\{6A_2G_+(G_+-3k^2)-3A_1[-2G_+{}^2+(2G_+-3k^2)(4+k^2)]\}}{[2k^4(G_+-G_-)(A_2-A_1)]},$$

$$C_3^2=-\varepsilon[6+(k^2-\tfrac{2}{3}G_-)C_1^2]/(k^2-\tfrac{2}{3}G_+),$$

$$C_2^2=k^2(C_1^2+C_3^2).$$

This solution is applicable in the region $1<\gamma\leqslant 2$, $0<k^2\leqslant 1$, or $A_3\geqslant 4A_2-3A_1$, and $A_2>A_1$. For $\varepsilon=+1$, $A_1>0$, and for $\varepsilon=-1$, $A_1,A_2,A_3<0$. It becomes Eq. (10) when $A_3=4A_2-3A_1$ for which $\gamma=2$, and $k^2=1$.

Solution (III):

$$x_1=C_1\alpha[\tfrac{1}{3}G_--k^2\mathrm{sn}^2(\alpha t,k)],$$

$$x_2=C_2\alpha k\,\mathrm{sn}(\alpha t)\mathrm{dn}(\alpha t,k),$$

$$x_3=C_3\alpha\left[\frac{1}{3}\,G_+-k^2\mathrm{sn}^2(\alpha t,k)\right],$$

where

$$G_{\pm}=1+k^2\pm(1-k^2+k^4)^{1/2},$$

$$\alpha^2=(A_3-A_2)/(1+4k^2-2G_-),$$

$$k^2=[\gamma^2-1+(3\gamma^2-3)^{1/2}]/(2\gamma^2-2),$$

$$\gamma=(A_3-A_1)/(A_3-2A_2+A_1),$$

$$C_1^2=\frac{\varepsilon\{6A_2G_+(G_+-3)-3A_1[-2G_+{}^2+(2G_+-3)(1+4k^2)]\}}{[2(G_+-G_-)(A_2-A_1)]},$$

$$C_3^2=-\varepsilon[6+(1-\tfrac{2}{3}G_-)C_1^2]/(1-\tfrac{2}{3}G_+),$$

$$C_2^2=C_1^2+C_3^2.$$

This solution is applicable in the region $\gamma\geqslant 2$, $1/2<k^2\leqslant 1$, or $A_1<A_2<A_3\leqslant 4A_2-3A_1$. For $\varepsilon=+1$, $A_1>0$, and for $\varepsilon=-1$, $A_1,A_2,A_3<0$. It becomes Eq. (10) when $A_3=4A_2-3A_1$ for which $\gamma=2$, and $k^2=1$.

These three solutions are examples that show that even though the solitary wave $\frac{1}{3}G_--k^2\mathrm{sn}^2(\alpha t,k)$ cannot propagate in the anomalous GVD region with another coupled wave, and the solitary wave $\frac{1}{3}G_+-k^2\mathrm{sn}^2(\alpha t,k)$ cannot propagate in the normal GVD region with another coupled wave, either of them *can* propagate in the normal *or* the anomalous GVD region if it is coupled with two other suitable solitary waves.

The periodic solutions are of increasing interest especially after a recent experimental observation of the evolution of an arbitrarily shaped input optical pulse train to the shape-preserving Jacobian elliptic pulse-train corresponding to the Maxwell-Bloch equations [13].

V. OTHER NONLINEAR EQUATIONS

It seems natural to ask whether the wave $\frac{1}{3}G_- - k^2\mathrm{sn}^2(\alpha t,k)$ or its aperiodic form $\mathrm{sech}^2(\alpha t)-\frac{2}{3}$ when $k^2=1$, appears as a solution of other simpler nonlinearly coupled dynamical equations. The answer is affirmative, and we shall give the following simple examples, even though the equations may not be of any great physical interest.

Consider the two coupled nonlinear equations given by

$$\ddot{x}_m+\varepsilon(x_1+x_2)x_m=A_m x_m, \quad m=1,2,$$
$$\text{and } \varepsilon=+1 \text{ or } -1. \tag{11}$$

These coupled equations may be considered as the associated dynamical equations of coupled equations of two interacting complex field components $\phi_1(z,t)$ and $\phi_2(z,t)$ that satisfy the following coupled equations:

$$i\phi_{mz}+\phi_{mtt}+\kappa_m\phi_m+\varepsilon(|\phi_1|+|\phi_2|)\phi_m=0, \quad m=1,2, \tag{12}$$

as the transformations (7) and (9) can be shown to apply to Eqs. (11) and (12) also. A solution of Eq. (11), for $A_2>A_1$, is

$$x_1=\varepsilon C_1\alpha^2[\tfrac{1}{3}G_- - k^2\mathrm{sn}^2(\alpha t,k)],$$
$$x_2=\varepsilon C_2\alpha^2[\tfrac{1}{3}G_+ - k^2\mathrm{sn}^2(\alpha t,k)],$$
$$\alpha^2=(A_2-A_1)/[2(G_+-G_-)],$$
$$G_\pm=1+k^2\pm(1-k^2+k^4)^{1/2}, \tag{13}$$
$$C_1=-6A_1/(A_2-A_1), \quad C_2=6A_2/(A_2-A_1), \quad 0<k^2\leqslant 1.$$

The aperiodic solution of Eq. (13) (for $k^2=1$) is

$$x_1=-\varepsilon\tfrac{3}{2}A_1 g_1, \quad x_2=\varepsilon\tfrac{3}{2}A_2 g_3,$$
$$\alpha^2=(A_2-A_1)/4. \tag{14}$$

The corresponding single nonlinear equation is

$$\ddot{x}+\varepsilon x^2=Ax. \tag{15}$$

For $A<0$, a solution of Eq. (15) is

$$x=6\varepsilon\alpha^2[\tfrac{1}{3}G_- - k^2\mathrm{sn}^2(\alpha t)], \tag{16a}$$
$$\alpha^2=-A/[2(G_+-G_-)], \quad G_\pm=1+k^2\pm(1-k^2+k^4)^{1/2},$$

which, for $k^2=1$, reduces to

$$x=-\varepsilon\tfrac{3}{2}Ag_1, \quad \alpha^2=-A/4. \tag{16b}$$

For $A>0$, a solution of Eq. (15) is

$$x=6\varepsilon\alpha^2[\tfrac{1}{3}G_+ - k^2\mathrm{sn}^2(\alpha t)], \tag{17a}$$
$$\alpha^2=A/[2(G_+-G_-)], \quad G_\pm=1+k^2\pm(1-k^2+k^4)^{1/2},$$

which, for $k^2=1$, reduces to

$$x=\tfrac{3}{2}A\varepsilon g_3, \quad \alpha^2=A/4. \tag{17b}$$

Note that solutions (16) and (17) for the single nonlinear equation (15) are exclusive of each other because of the condition that A is <0 or >0, but the two coupled equations (11) bring them together as solutions for x_1 and x_2, respectively, the required condition being simply $A_1\neq A_2$ (we have assumed $A_1<A_2$ in our solutions (13) and (14) but the order can be clearly interchanged). This is analogous to the situation we found when we considered the solutions from $N=1$ to $N=2$ to $N=3$ for Eqs. (8), which we discussed following Eq. (10). Notice that the red-blue (g_1,g_3) combination given by Eqs. (14) is not found for Eqs. (8) for $N=2$.

VI. SUMMARY

In summary, we have presented solitary waves for two and three coupled NLS equations, and in particular, solitary waves [(i)–(v) in Sec. III and solution (3) of Table II] for $N=2$ that can propagate in the normal GVD region and, when coupled with a third solitary wave, can propagate in either the normal or the anomalous GVD region [Eq. (10), solutions (I)–(III) in Sec. IV, and solutions (4) and (6) in Tables IV and VI]. The wave $\frac{1}{3}G_- - k^2\mathrm{sn}^2(\alpha t,k)$ or its aperiodic form $\mathrm{sech}^2(\alpha t)-\frac{2}{3}$ is shown to be a solution of other nonlinear equations (Sec. V) and is thus not uncommon. These solitary waves are stable for $\varepsilon=+1$, at least linearly stable, as can be shown by following the stability analysis given by Infeld [14] for similar periodic and aperiodic coupled waves given by Grobe and the author [15]. The special feature of this result is that not only a wave form g_1 of solitary wave has been found, but also the introduction of the idea that (i) a second generation of solitary waves (g_1,g_2,g_3) which form the simplest set of three different or complementary waves, may, in addition to the two solitary waves (f_1,f_2) of the first generation, become a useful and practical tool, and (ii) a third coupled wave may indeed be helpful for extending the region of applicability for propagation of a pair of solitary waves. Idea (i) may be used for extending the variational approach [16] and may stimulate systematic searches for the next generation of solitary waves. Idea (ii) gives a concrete example that extended the very successful idea of using two optical waves instead of one for better control of wave propagation [15,17,18] to using three optical waves instead of two.

ACKNOWLEDGMENT

This research was supported by the NSF under Grant No. PHY-9507837.

[1] See, e.g., G. P. Agrawal, *Nonlinear Fiber Optics* (Academic, New York, 1995), and the references cited therein.
[2] C. R. Menyuk, Opt. Lett. **12**, 614 (1987); J. Opt. Soc. Am. B **5**, 392 (1988).
[3] V. V. Afanasyev, Y. S. Kivshar, V. V. Konotop, and V. N. Serkin, Opt. Lett. **14**, 805 (1989).
[4] S. Trillo, S. Wabnitz, E. M. Wright, and G. I. Stegeman, Opt. Lett. **13**, 871 (1988).
[5] Y. S. Kivshar and S. K. Turitsyn, Opt. Lett. **18**, 337 (1993); A. P. Sheppard and Y. S. Kivshar, Phys. Rev. E **55**, 4773 (1997).
[6] M. V. Tratnik and J. E. Sipe, Phys. Rev. A **38**, 2011 (1988).
[7] Y. Silbergerg and Y. Barad, Opt. Lett. **20**, 246 (1995).
[8] D. N. Christodoulides and R. I. Joseph, Opt. Lett. **13**, 53 (1988).
[9] F. T. Hioe, Phys. Lett. A **234**, 351 (1997).
[10] M. Florjanczyk and R. Tremblay, Phys. Lett. A **141**, 34 (1989).
[11] N. A. Kostov and I. M. Uzunov, Opt. Commun. **89**, 389 (1992).
[12] F. T. Hioe, Phys. Rev. E **56**, 7253 (1997); **58**, 1174 (1998).
[13] J. L. Shultz and G. J. Salamo, Phys. Rev. Lett. **78**, 855 (1997).
[14] E. Infeld, Phys. Rev. Lett. **75**, 1419 (1995).
[15] F. T. Hioe and R. Grobe, Phys. Rev. Lett. **73**, 2559 (1994).
[16] C. Pare, Phys. Rev. E **54**, 846 (1996).
[17] R. Grobe, F. T. Hioe, and J. H. Eberly, Phys. Rev. Lett. **73**, 3183 (1994); F. T. Hioe and C. E. Carroll, Phys. Rev. A **56**, 2292 (1997).
[18] See, e.g., S. E. Harris, Phys. Today **50** (7), 36 (1997), and many references cited therein.