9-1997

Solitary waves and atomic population transfer via the continuum

Foek T. Hioe
Saint John Fisher College, fhioe@sjfc.edu

C. E. Carroll
University of Rochester

Publication Information

Hioe, Foek T. and Carroll, C. E. (1997). "Solitary waves and atomic population transfer via the continuum." *Physical Review A* 56.3, 2292-2298.

Exact analytic results are presented that represent a pair of solitary waves that can propagate through an atomic medium with their shapes invariant while producing transfer of atoms or molecules from one bound state to another via a continuum. The speed of the waves and the amount of population transfer are given in terms of the wave amplitudes. A general ~not solitary wave! solution for the case of a single wave is presented, together with a nonlinear Beer's law of absorption.

Solitary waves and atomic population transfer via the continuum

F. T. Hioe and C. E. Carroll*
Department of Physics, St. John Fisher College, Rochester, New York 14618
(Received 8 April 1997)

Exact analytic results are presented that represent a pair of solitary waves that can propagate through an atomic medium with their shapes invariant while producing transfer of atoms or molecules from one bound state to another via a continuum. The speed of the waves and the amount of population transfer are given in terms of the wave amplitudes. A general (not solitary wave) solution for the case of a single wave is presented, together with a nonlinear Beer's law of absorption. [S1050-2947(97)04309-6]

PACS number(s): 42.50.Gy, 42.65.Tg, 03.40.Kf

I. INTRODUCTION

The problem of rendering a multilevel optical medium transparent and of controlling its refractive index has been of considerable interest for a number of years. The early works of Konopnicki, Drummond, and Eberly [1] and of Hioe [2] introduced simultaneous multiple solitary waves of possibly different wavelengths but necessarily of similar shapes. Following the recent work of Harris and co-workers [3,4] on electromagnetically induced transparency, in which an approximate transparency for the probe laser pulse is obtained by applying a strong pulse at the Stokes wavelength in lambda-type three-level medium, a greater understanding was achieved when analytic results were obtained [5,6] that give the exact and approximately matched solitary waves of similar or very different shapes that can propagate through a lambda-type three-level medium with their shapes invariant. These results received further support from observation [7] of the theoretically predicted ''adiabatons,'' which can propagate without attenuation at a speed that is only a fraction of the speed of light.

The numerous experimental successes [8] that verify the concept of adiabatic population transfer from one bound state of an atom or molecule to another in a Raman-type transition driven by two laser pulses arranged in the counterintuitive order [9] have led us to suggest [10,11] that the intermediate bound state can be replaced by a narrow region of the continuum. We have shown this to be possible first for a flat and featureless continuum, and more recently for a continuum with some structure, which determines the pulse delay, laser frequencies, and laser intensities that give optimum population transfer. Whether the intermediary consists of many energy levels or a continuum, pulse propagation through a medium of such systems is relevant to studies of optical pulse propagation in nonlinear media.

We present in Sec. II *exact* analytic results for a pair of solitary waves that propagate with their shapes invariant through an atomic medium; they involve population transfer between two bound states with a continuum as intermediary. Although we assume a flat and featureless continuum, these analytic results give useful insights on the type of pulse shape needed for shape-invariant propagation and on how the pulse amplitudes affect the speed of solitary waves. These results may suggest, for example, how the refractive index of the atomic medium can be controlled.

We present in Sec. III a *general* analytic solution for the case of population transfer from a single bound state to a continuum, and present a general Beer's absorption law applicable to any pulse amplitude. This general solution shows clearly how a solitary wave compares with propagation of an external pulse into a medium.

II. TWO BOUND STATES AND A CONTINUUM

For each atom or molecule in a medium, let A_1 and A_2 be the probability amplitudes for occupation of bound states 1 and 2 with (negative) energies E_1 and E_2. Two light waves with carrier frequencies ω_1 and ω_2 propagating in the medium can drive transitions from one bound state to the other; these are stimulated Raman transitions through continuum states near the positive energy E, where $\omega_1=(E+|E_1|)/\hbar$ and $\omega_2=(E+|E_2|)/\hbar$. We shall write equations of motion for this process. Direct transitions between the two bound states are forbidden or negligible. Transitions from either bound state to the continuum occur at a rate proportional to the light wave intensity and to the square of the transition matrix element. When both light waves are present, cross terms proportional to the geometric-mean intensity occur. Distant parts of the continuum give rise to an ac Stark shift of each bound state relative to the bottom of the continuum. Such results are obtained by starting with the Schrödinger equation for an atom or molecule driven by an external electric field. We compute the atomic polarization and put it into Maxwell's equations. Following Knight, Lauder, and Dalton [12], adiabatic elimination of all continuum states is used; it can be justified if the ''width'' of the continuum times the laser pulse duration is large compared to Planck's constant. This computation is not applicable to a continuum with a narrow autoionizing state. It gives the rates of change of A_1 and A_2 as linear functions of the two light wave intensities and their geometric mean. Also, it gives the rates of change of $\mathcal{E}_1$ and $\mathcal{E}_2$, the two electric-field envelopes, as linear functions of $|A_1|^2$, $|A_2|^2$, $A_1^*A_2$, and $A_1A_2^*$. We assume that the light wave frequencies are chosen so that unwanted transitions to distant parts of the continuum, transitions from state 1 (2) to the continuum driven by light wave 2 (1), are neg-

*Also at Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627.

ligible [11]. Also, we use the one-photon resonance condition $E_1+\hbar\omega_1=E_2+\hbar\omega_2=E$. This computation give us four coupled nonlinear partial differential equations for A_1, A_2, $\mathcal{E}_1$, and $\mathcal{E}_2$. Instead of $\mathcal{E}_1$ and $\mathcal{E}_2$, we shall use two dimensionless electric-field amplitudes defined by

$$\Omega_1=\left(\frac{\ell_2}{\ell_1}\right)^{1/4}\frac{\mathcal{E}_1}{(2\pi n\hbar\omega_1)^{1/2}},\quad \Omega_2=\left(\frac{\ell_1}{\ell_2}\right)^{1/4}\frac{\mathcal{E}_2}{(2\pi n\hbar\omega_2)^{1/2}}. \tag{1}$$

Here, n is the number of atoms or molecules per unit volume, ℓ_1 and ℓ_2 are the Beer's lengths [13] given by

$$1/\ell_j=4\pi^2 n d_j^2\rho(E)\omega_j/c \quad \text{for } j=1 \text{ or } 2, \tag{2}$$

where c is the speed of light, d_j is the electric-dipole matrix element for transitions from bound state j to the continuum at energy E, where the density of states is $\rho(E)$. Note that

$$\frac{|\mathcal{E}_1|^2}{2\pi n\hbar\omega_1}+\frac{|\mathcal{E}_2|^2}{2\pi n\hbar\omega_2}$$

is the number of photons per atom or molecule. The four coupled equations for A_1, A_2,Ω_1, and Ω_2 are

$$\frac{\partial A_1}{\partial t}+\frac{c}{2(\ell_1\ell_2)^{1/2}}\Omega_1^*[(1-iq_1)A_1\Omega_1+(1-iq_{12})A_2\Omega_2]=0, \tag{3a}$$

$$\frac{\partial A_2}{\partial t}+\frac{c}{2(\ell_1\ell_2)^{1/2}}\Omega_2^*[(1-iq_{12})A_1\Omega_1+(1-iq_2)A_2\Omega_2]=0, \tag{3b}$$

$$\frac{\partial\Omega_1}{\partial z}+\frac{1}{c}\frac{\partial\Omega_1}{\partial t}+\frac{1}{2\ell_1}A_1^*[(1-iq_1)A_1\Omega_1+(1-iq_{12})A_2\Omega_2]=0, \tag{3c}$$

and

$$\frac{\partial\Omega_2}{\partial z}+\frac{1}{c}\frac{\partial\Omega_2}{\partial t}+\frac{1}{2\ell_2}A_2^*[(1-iq_{12})A_1\Omega_1+(1-iq_2)A_2\Omega_2]=0, \tag{3d}$$

where the continuum structure and E determine the dimensionless parameters q_1, q_2, and q_{12}. Each piece of the electric field has been expressed in terms of a polarization unit vector, a phase $\exp(ik_jz-i\omega_jt)$, and a slowly varying amplitude $\mathcal{E}_j(z,t)$. We have assumed that $\omega_1/k_1=\omega_2/k_2=c$. For solitary waves that are shape invariant and that propagate through the medium with velocity v, the variables depend on t and z through $\xi=(t-z/v)/\tau$,where the (positive) pulse length τ and velocity v depend on the pulse amplitudes, as shown below. We assume that the continuum is flat, so that $q_1=q_2=q_{12}=0$. We also assume that $\ell_1=\ell_2=\ell$, a condition that places some restriction on the choice of the laser frequencies ω_1 and ω_2;see Eq. (2).

We have found exact analytic solitary-wave solutions of the coupled nonlinear Schrödinger-Maxwell equations (3). They are

$$A_1(\xi)=\beta^{-1}[Af_2(\xi)-Bf_1(\xi)],$$
$$A_2(\xi)=\beta^{-1}[Cf_2(\xi)-Df_1(\xi)], \tag{4a}$$

with

$$\Omega_1(\xi)=Af_1(\xi)+Bf_2(\xi),\quad \Omega_2(\xi)=Cf_1(\xi)+Df_2(\xi), \tag{4b}$$

where A, B, C, and D are arbitrary real constants, β is a normalization constant, and f_1 and f_2 are functions to be given shortly. First, we define the following combinations of A, B, C, and D:

$$a=AB+CD,\quad b=\tfrac{1}{2}(A^2-B^2+C^2-D^2). \tag{5}$$

In terms of these constants,

$$f_1(\xi)=\pm\frac{1}{\sqrt{2}}\left(1+\frac{b}{\sqrt{a^2+b^2}}\tanh\xi-\frac{a}{\sqrt{a^2+b^2}}\operatorname{sech}\xi\right)^{1/2}, \tag{6a}$$

and

$$f_2(\xi)=\pm\frac{1}{\sqrt{2}}\left(1-\frac{b}{\sqrt{a^2+b^2}}\tanh\xi+\frac{a}{\sqrt{a^2+b^2}}\operatorname{sech}\xi\right)^{1/2}, \tag{6b}$$

where the initial value of ξ is arbitrary, but we shall use $-\infty$ as the initial value. The pulse length τ and the velocity v of the solitary wave are related to the pulse amplitudes A, B, C, and D by

$$\tau=(a^2+b^2)^{-1/2}\ell/c \tag{7}$$

and

$$v=c/(1+\beta^{-2}). \tag{8}$$

We choose initial value that satisfy $|A_1(-\infty)|^2+|A_2(-\infty)|^2=1$; this implies that β is given by

$$\beta^2=\tfrac{1}{2}(A^2+B^2+C^2+D^2)+\sqrt{a^2+b^2}. \tag{9}$$

The functions f_1 and f_2 in Eq. (6) can be positive or negative. To determine the ambiguous signs, let

$$f_1(\xi)=\sin\varphi \quad \text{and } f_2(\xi)=\cos\varphi, \tag{10}$$

where φ is determined by

$$\sin 2\varphi=\frac{1}{\sqrt{a^2+b^2}}(a\tanh\xi+b\operatorname{sech}\xi),$$

$$\cos 2\varphi=\frac{1}{\sqrt{a^2+b^2}}(-b\tanh\xi+a\operatorname{sech}\xi). \tag{11}$$

Since $\sin 2\varphi=2f_1f_2$and $\cos 2\varphi=f_2^2-f_1^2$, this determines the relative signs of f_1 and f_2 in Eq. (4).

Equations (4)–(9), with the clarification given by Eqs. (10) and (11), complete the description of our solitary-wave solution of the nonlinear coupled equations (3). They can be verified by direct substitution. Three general remarks can be made before we consider special cases. First, the combination of dynamic variables $\Omega_2(\xi)A_1(\xi)-\Omega_1(\xi)A_2(\xi)$ is a constant of the motion; it is equal to $\beta^{-1}(AD-BC)$. Second, this solitary pulse pair does not return the atoms or molecules in the medium to their initial states. Third, this solution allows various initial states for the atoms or molecules, depending on the choices of the arbitrary constants A, B, C, and D and the initial value of ξ. The atoms or molecules are all initially (at $\xi=-\infty$) in their ground states if we choose $B=C=0$ and $|A|>|D|$ or choose $A=D=0$ and $|C|>|B|$.

We now consider some special cases, assuming that the values of ξ run from $-\infty$ to $+\infty$:

A. Special case of $B=C=0$, with $|A|>|D|$

$$\Omega_1=\frac{A}{\sqrt{2}}(1+\tanh\xi)^{1/2},\quad \Omega_2=\frac{D}{\sqrt{2}}(1-\tanh\xi)^{1/2}, \tag{12a}$$

$$A_1=\frac{1}{\sqrt{2}}(1-\tanh\xi)^{1/2},\quad A_2=\frac{-D}{\sqrt{2}A}(1+\tanh\xi)^{1/2}, \tag{12b}$$

$$\tau=\frac{\ell}{c}\frac{2}{A^2-D^2},\quad \text{and}\quad v=\frac{c}{1+A^{-2}}.$$

This is a case of two pulses which are ''complementary'' to each other and are incident in the counterintuitive order. The atoms or molecules are initially in their ground state $[|A_1(-\infty)|^2=1$ and $|A_2(-\infty)|^2=0]$, and the final occupation probabilities are $|A_1(+\infty)|^2=0$ and $|A_2(+\infty)|^2=D^2/A^2$. Transfer to the continuum accounts for the remaining probability. The case of $A=D=0$, $|B|>|C|$ is similar.

B. Special case of $B=D=0$

$$\Omega_1=\frac{A}{\sqrt{2}}(1+\tanh\xi)^{1/2},\quad \Omega_2=\frac{C}{\sqrt{2}}(1+\tanh\xi)^{1/2}, \tag{13a}$$

$$A_1=\frac{A}{\sqrt{2}\beta}(1-\tanh\xi)^{1/2},\quad A_2=\frac{C}{\sqrt{2}\beta}(1-\tanh\xi)^{1/2} \tag{13b}$$

$$\beta=(A^2+C^2)^{1/2},$$

$$\tau=\frac{\ell}{c}\frac{2}{A^2+C^2},\quad \text{and}\quad v=\frac{c}{1+(A^2+C^2)^{-1}}.$$

This is a case where the two pulses are similar in shape but have possibly different amplitudes A and C. The two bound states of the atoms or molecules in the medium are initially populated in the ratio of A^2 to C^2, and they finally lose all their population to the continuum.

C. Special case of $A_2=\mathcal{E}_2=0$ or $C=D=0$

$$\Omega_1(\xi)=Af_1(\xi)+Bf_2(\xi)$$

$$\text{and}\quad A_1(\xi)=\beta^{-1}[Af_2(\xi)-Bf_1(\xi)]. \tag{14}$$

Here, f_1 and f_2 are given by Eq. (6), $a=AB, b=(A^2-B^2)/2$, $\beta=(A^2+B^2)^{1/2}$,

$$\tau=\frac{\ell}{c}\frac{2}{A^2+B^2}\quad \text{and}\quad v=\frac{c}{1+(A^2+B^2)^{-1}}.$$

This is the case in which one light wave has negligible intensity. Thus, only one bound state (state 1) and the continuum are involved. We note that the squares of Ω_1 and A_1 are simply

$$|\Omega_1|^2=\tfrac{1}{2}(A^2+B^2)(1+\tanh\xi)\quad \text{and}\quad |A_1|^2=\tfrac{1}{2}(1-\tanh\xi). \tag{15}$$

Here, we have written the absolute squares of Ω_1 and A_1, because Eq. (15) is valid even when Ω_1 and A_1 are complex. In that case, the two equations in Eq. (3) that are not trivially satisfied can be expressed as coupled differential equations for $|\Omega_1|^2$ and $|A_1|^2$. The dependence on q_1 disappears, without assuming that $q_1=0$. Thus Eq. (15) is a solitary-wave solution for the case of one bound state, even when the continuum structure is significant and when Ω_1 and A_1 are complex. Indeed, we have found a *general* solution of the coupled differential equations for $|\Omega_1|^2$ and $|A_1|^2$; the solitary wave (15) is a special case. This general solution is treated in Sec. III.

There is another type of solitary wave that can propagate through the medium with invariant shape, and it is quite distinct from Eq. (4). This solitary-wave solution exists under less restrictive conditions than Eq. (4): First, we need to assume $q_1=q_2=q_{12}$, but these parameters need not vanish. Second, we need not assume $\ell_1=\ell_2$. Third, we need not assume two-photon resonance; Eqs. (3a) and (3b) may contain detuning terms $-i(E-E_j-\hbar\omega_j)A_j/\hbar$, where $j=1$ and 2. Fourth, we need not assume that the phase velocities are equal to c. If the phase velocities are $\omega_1/k_1=u_1$ and $\omega_2/k_2=u_2$, the first two terms in Eqs. (3c) and (3d) are replaced by $[\partial/\partial z+\partial/\partial(u_jt)]\Omega_j-i\omega_j(1-u_j^2/c^2)\Omega_j/(2u_j)$, where $j=1$ or 2. It is not hard to solve Eq. (3) even with the above additional terms; we find a solution in which $|A_1|^2$, $|A_2|^2$, $|\Omega_1|^2$, and $|\Omega_2|^2$ are constants and $A_1\Omega_1+A_2\Omega_2=0$. Assuming that $u_1=u_2=c$, we can generalize this solution by considering functions A_1, A_2, Ω_1, and Ω_2 that depend on $\xi=(t-z/c)/\tau$. It is easy to see that if the initial values of A_1, A_2, Ω_1, and Ω_2 are such that $A_1\Omega_1+A_2\Omega_2=0$, or

$$\Omega_1/\Omega_2=-A_2/A_1, \tag{16}$$

then the atoms or molecules in the medium do not evolve. To keep them from evolving, the shapes or the ξ dependence of Ω_1 and Ω_2 can be arbitrary, but the ratio Ω_1/Ω_2 must be kept constant and equal to its initial value, assuming that the probability amplitudes of the bound states initially satisfy Eq. (16). These probability amplitudes do not change as the pulse pair propagates through the medium with speed c. If all the atoms or molecules are initially in the ground state and if we start with $|\Omega_2|\gg|\Omega_1|$, the leading edges of the propagating pulses will adjust the populations of the two bound states so that Eq. (16) is approximately satisfied, thus providing the basis for the ''electromagnetically induced transparency,'' as treated in a medium of three-state atoms by Harris and coworkers [3,4]. We call the solitary waves that satisfy Eq. (16) the ''dark-state'' solitary waves. They are quite distinct from our solitary waves (4), which generally involve significant transfer of population among two bound states and the continuum and which propagate with a speed that is less than c and that depends on the amplitudes. For the three-state medium solitary waves that satisfy Eq. (16) were found by Rahman, Grobe, and Eberly [14]. For certain closed-loop systems, these solitary waves were found by the present authors [15].

To see the relation of the system of two bound states and a flat continuum to a lambda-type three-level system, we compare Eq. (3) with the equations of motion for a lambda-type system. The Schrödinger equation gives

$$i\frac{\partial A_j}{\partial t}=-\frac{1}{2}\overline{\Omega}_j^* A_m \quad \text{for } j=1 \text{ or } 2, \tag{17a}$$

$$i\frac{\partial A_m}{\partial t}=-\frac{1}{2}\overline{\Omega}_1 A_1-i\gamma A_m-\frac{1}{2}\overline{\Omega}_2 A_2, \tag{17b}$$

where A_m is the probability amplitude for the high-lying intermediate level, γ is half the decay rate from that level, and $\overline{\Omega}_j=2d_j\mathcal{E}_j/\hbar$ is the Rabi frequency for transitions between level j and the intermediate level m. The Maxwell equations give

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\overline{\Omega}_1=2i\mu_1 A_1^* A_m \tag{17c}$$

and

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\overline{\Omega}_2=2i\mu_2 A_m A_2^*, \tag{17d}$$

where the propagation constant is $\mu_j=2\pi n d_j^2\omega_j/\hbar c$. A formal mapping of Eq. (17) into Eq. (3) can be obtained by using the adiabatic approximation; we assume $\partial A_m/\partial t=0$, find $A_m=(i/2\gamma)(\overline{\Omega}_1 A_1+\overline{\Omega}_2 A_2)$, and substitute this into Eqs. (17a), (17c), and (17d). This mapping is completed by changing $(\gamma\hbar)^{-1}$ to $\rho(E)$; see Eq. (2).

It would be interesting to compare our solitary-wave solution given by Eqs. (6) and (10) with the solitary-wave solutions for a lambda-type three-level medium that are given in Table I of [5]. The equations treated in [5] are similar to Eq. (17), but $\gamma=0$ and $\partial A_m/\partial t\neq 0$.

We note that the four coupled equations (3), when written in terms of a single independent variable ξ, are mathematically similar to the nonlinear equations for four-wave interactions in photorefractive media [16,17]. However, neither $|A_1|^2+|A_2|^2$ nor $|\Omega_1|^2+|\Omega_2|^2$ is a conserved quantity in our solution of Eq. (3). The solution given by Zozulya and Tikhonchuk [17] is therefore somewhat different from our solution. Furthermore, it does not contain the arbitrary amplitudes A, B, C, and D which are a central feature in our solution. They can be used to control wave propagation.

III. ONE BOUND STATE AND A CONTINUUM

In this section, we present an analytic general solution of the coupled Schrödinger and Maxwell wave equations for a laser pulse that propagates through an atomic medium and causes transitions from one bound state to the continuum. This solution shows how the shape and speed of the pulse change with propagation distance and it gives a nonlinear Beer's law of absorption. An analytic solitary-wave solution is also presented.

The problem of pulse propagation through a two-level atomic medium was studied many years ago [13]. One familiar result is the exponential decay of the pulse amplitude or pulse energy as distance increases. This decay law, which follows from a linear differential equation, is applicable for small pulse amplitudes. Another familiar solution is the hyperbolic-secant pulse of area 2π, which is found when the upper level does not decay. The coupled Schrödinger and Maxwell equations for pulse propagation are nonlinear partial differential equations. The closely related problem in which the upper level is replaced by a continuum is much less studied. Study of this problem is motivated by our recent results that a continuum can be used as an intermediary for population transfer from one bound state to another [10,11] and that it is possible for a solitary wave to propagate while inducing bound-continuum-bound transitions in the atoms or molecules of the medium, as shown in Sec. II.

In this section, we present the *general* analytic solution of the coupled Schrödinger and Maxwell equations (3) for the case of propagation of a *single* wave that transfers the atomic population from a bound state to a continuum. This solution gives us the generalization of Beer's absorption law to arbitrary pulse amplitudes. Also, it enables us to see how the pulse shape and speed change as the pulse penetrates into the medium. The relationship of the bound-continuum model to the two-level model will also be given.

Setting A_2 and Ω_2 equal to zero and dropping the subscripts from A_1, Ω_1, q_1, and ℓ_1, the coupled Schrödinger and Maxwell equations give

$$\frac{\partial A}{\partial t}+\frac{c(1-iq)}{2\ell}A|\Omega|^2=0 \tag{18a}$$

and

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\Omega+\frac{(1-iq)}{\ell}|A^2|\Omega=0, \tag{18b}$$

where q is the parameter that represents continuum structure and the ac Stark shift. From Eq. (18) and the complex-conjugate equations, it is easy to get

$$\frac{\partial|A|^2}{\partial t}+\frac{c}{\ell}|A|^2|\Omega|^2=0$$

$$\text{and}\quad \left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)|\Omega|^2+\frac{1}{\ell}|A|^2|\Omega|^2=0, \tag{19}$$

where q does not appear. We shall use the dimensionless position and time

$$\overline{z}=z/\ell \quad \text{and} \quad \overline{t}=(c/\ell)(t-z/c). \tag{20}$$

Using these independent variables, Eq. (19) becomes

$$\frac{\partial}{\partial \bar{t}}|A|^2+|A|^2|\Omega|^2=0 \quad \text{and} \quad \frac{\partial}{\partial \bar{z}}|\Omega|^2+|A|^2|\Omega|^2=0. \tag{21}$$

The general solution of Eq. (21) can be written in analytic form. This is indicated by the test of Weiss, Tabor, and Carnevale [18]. They apply the Painlevé criterion to nonlinear partial differential equations. This is done by putting the curve $\phi(\bar{z},\bar{t})=0$ in place of the usual point singularity and then constructing ascending power series that start with a negative power of ϕ. In the application to Eq. (21), the series for $|A|^2$ and $|\Omega|^2$ start with terms in $1/\phi$, and the conclusion is that Eq. (21) is integrable. The formal solution constructed in this way can be truncated and simplified by assuming that $\partial^2\phi/\partial\bar{z}\partial\bar{t}=0$. Each series reduces to one term. We obtain

$$|A(\bar{z},\bar{t})|^2=\frac{F'(\bar{z})}{F(\bar{z})+G(\bar{t})}=\frac{\partial}{\partial \bar{z}}\ln[F(\bar{z})+G(\bar{t})] \tag{22a}$$

and

$$|\Omega(\bar{z},\bar{t})|^2=\frac{G'(\bar{t})}{F(\bar{z})+G(\bar{t})}=\frac{\partial}{\partial \bar{t}}\ln[F(\bar{z})+G(\bar{t})], \tag{22b}$$

which contain two arbitrary functions $F(\bar{z})$ and $G(\bar{t})$. Since the arbitrary functions can be used to satisfy boundary conditions on the lines $\bar{z}=\bar{z}_0$ and $\bar{t}=\bar{t}_0$, we have obtained the general solution of Eq. (21). The phases of A and Ω can be obtained from Eq. (18).

The usual initial condition is that every atom or molecule is in the bound state before the optical pulse arrives, so that $|A|^2=1$ initially. Assuming this, we find $F(\bar{z})=\exp\bar{z}$ and $G(\bar{t}_i)=0$, which give

$$|A(\bar{z},\bar{t})|^2=\frac{\exp\bar{z}}{-1+\exp\bar{z}+\exp\int_{\bar{t}_i}^{\bar{t}}|\Omega(0,\bar{t}')|^2 d\bar{t}'} \tag{23a}$$

and

$$|\Omega(\bar{z},\bar{t})|^2=\frac{|\Omega(0,\bar{t})|^2\exp\int_{\bar{t}_i}^{\bar{t}}|\Omega(0,\bar{t}')|^2 d\bar{t}'}{-1+\exp\bar{z}+\exp\int_{\bar{t}_i}^{\bar{t}}|\Omega(0,\bar{t}')|^2 d\bar{t}'} \tag{23b}$$

The initial dimensionless time $\bar{t}_i$ is ordinarily $-\infty$. The pulse shape at zero depth is $\Omega(0,\bar{t})$, an arbitrary function of $\bar{t}$.

An alternative solution of Eq. (21) uses a change of variables involving the exponential function to get Liouville's nonlinear partial differential equation [19]. This solution was published by Brekhovskikh *et al.* [20], and similar calculations are given by Okladnikov *et al.* [21]. These authors obtained equations equivalent to Eq. (23), but they did not discuss the following physical features of the solution.

We define the dimensionless pulse energy as

$$E_p(\bar{z})\equiv\int_{-\infty}^{+\infty}|\Omega(\bar{z},\bar{t})|^2 d\bar{t}. \tag{24}$$

We use Eqs. (22b) and (23b) to obtain

$$E_p(\bar{z})=\ln\{1+[-1+\exp E_p(0)]\exp(-\bar{z})\}. \tag{25}$$

For $E_p(0)\ll 1$, we have the exponential decay law

$$E_p(\bar{z})\approx E_p(0)\exp(-\bar{z}). \tag{26}$$

For $E_p(0)\gg 1$, we can obtain simple approximate formulas applicable to small and large $\bar{z}$. For $E_p(0)-\bar{z}\gg 1$, we have linear decay:

$$E_p(\bar{z})\approx E_p(0)-\bar{z}. \tag{27}$$

For $\bar{z}-E_p(0)\gg 1$, we have exponential decay:

$$E_p(\bar{z})\approx[\exp E_p(0)]\exp(-\bar{z}). \tag{28}$$

Equation (25) is our nonlinear Beer's law. It is unlike the familiar Beer's law in that the differential equations governing the variation of pulse energy with distance are nonlinear. The velocity with which the energy of the pulse propagates through the atomic medium can be found by first determining $d\bar{t}_c(\bar{z})/d\bar{z}$, where $\bar{t}_c(\bar{z})$ is the centroid of the area under the curve $|\Omega(\bar{z},\bar{t})|^2$, meaning that

$$\bar{t}_c(\bar{z})=\frac{\int_{-\infty}^{+\infty}\bar{t}|\Omega(\bar{z},\bar{t})|^2 d\bar{t}}{\int_{-\infty}^{+\infty}|\Omega(\bar{z},\bar{t})|^2 d\bar{t}}. \tag{29}$$

The velocity of the pulse energy is

$$v(\bar{z})=\frac{c}{1+(d\bar{t}_c/d\bar{z})}$$

The peak position $\bar{t}_p$ of the pulse could be used to compute

$$v_p(\bar{z})=\frac{c}{1+(d\bar{t}_p/d\bar{z})},$$

the velocity of the peak.

The foregoing analytic results are used to generate Fig. 1, which shows changes in pulse shape and amplitude for an initially (at $\bar{z}=0$) Gaussian pulse; we use $|\Omega(0,\bar{t})|^2=\exp(-\bar{t}^2/2)$. The centroid and the peak move to larger values of $\bar{t}$ as $\bar{z}$ increases, because the leading edge is attenuated much more than the trailing edge. This is easily seen from Eq. (23b), which give $|\Omega(\bar{z})|^2\approx|\Omega(0)|^2\exp(-\bar{z})$ for

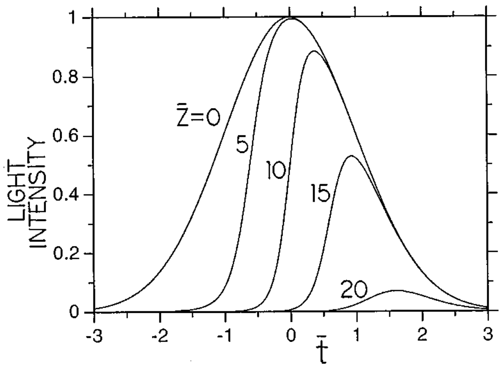

FIG. 1. Ratio of light intensity to its maximum value at $\bar{z}=0$, for $E_p(0)=20$, as a function of dimensionless time $\bar{t}$, for $\bar{z}=0$, 5, 10, 15, and 20.

the leading edge, and $|\bar{\Omega}(\bar{z})|^2 \approx |\bar{\Omega}(0)|^2 \exp[E_p(0)]/\{-1+\exp(\bar{z})+\exp[E_p(0)]\}$ for the trailing edge. If the pulse energy were to propagate at velocity c, the centroid of the area under the curve $|\bar{\Omega}(\bar{z})|^2$ would be at $\bar{t}=0$. However, the centroid is clearly seen to move to the right as $\bar{z}$ increases, so that $d\bar{t}_c(\bar{z})/d\bar{z}>0$, meaning that the pulse energy propagates at a velocity less than c. After the pulse is somewhat attenuated, its velocity starts to increase, finally approaching c. Also, attenuation of the leading edge can make the pulse shape asymmetric; see Fig. 1.

To see the relation between the present medium, in which each atom has one bound state and continuum, and the medium of two-level atoms, we compare Eq. (18a) with the rates of change obtained from the Schrödinger equation:

$$i\frac{\partial A_1}{\partial t} = -\frac{1}{2}\bar{\Omega}^* A_2 \tag{30a}$$

and

$$i\frac{\partial A_2}{\partial t} = -\frac{1}{2}\bar{\Omega} A_1 - (\delta + i\gamma)A_2, \tag{30b}$$

where δ and γ are the detuning and half the decay rate of level two, the upper level. Also, $\bar{\Omega}=2d\mathcal{E}/\hbar$ is the Rabi frequency for this transition; the Maxwell equations are used to show that it satisfies

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\bar{\Omega} = 2i\mu A_1^* A_2, \tag{30c}$$

where $\mu = 2\pi n d\omega/\hbar c$. A formal mapping from Eqs. (30) to (18) can be obtained if we start with the adiabatic approximation. From $\partial A_2/\partial t=0$, we obtain $A_2=-[2(\delta + i\gamma)]^{-1}\bar{\Omega}A_1$. This is substituted into Eqs. (30a) and (30c). Finally, we identify $[\gamma\hbar(1+\delta^2/\gamma^2)]^{-1}$ and $-\delta/\gamma$ as the quantities that should replace $\rho(E)$ and q in Eqs. (2) and (18), respectively.

The solitary-wave solution of Eq. (19) can be easily obtained. This solitary wave propagates through the medium with invariant shape, and with velocity v. Assuming that $|\Omega|^2$ and $|A|^2$ depend only on

$$\xi = (t - z/v)/\tau, \tag{31}$$

we find

$$|\Omega(\xi)|^2 = \tfrac{1}{2}a^2(1+\tanh\xi) \quad \text{and} \quad |A(\xi)|^2 = \tfrac{1}{2}(1-\tanh\xi), \tag{32}$$

where

$$\tau = \frac{2\ell}{a^2 c} \quad \text{and} \quad v = \frac{c}{1+a^{-2}}$$

are the pulse duration and velocity. Here, the atomic population is initially (at $\xi=-\infty$) in the bound state; it is completely transferred to the continuum at the end of the pulse ($\xi=+\infty$). This is possible because Eq. (24) is infinite here. The solitary wave (32) should be compared with the well-known hyperbolic-secant pulse of McCall and Hahn [22]. Their theory uses a medium of two-level atoms, and the equations of motion are similar to Eq. (30), but with $\gamma=0$ and $\partial A_2/\partial t \neq 0$.

Finally, we connect the solitary wave (32) with the general solution (22). We write Eq. (31) as

$$\xi = \frac{\ell}{c\tau}\bar{t} - \left(\frac{1}{v} - \frac{1}{c}\right)\frac{\ell}{\tau}\bar{z},$$

and use this to put Eq. (32) into the form (22), with

$$F(\bar{z}) = \exp\left[2\left(\frac{1}{v}-\frac{1}{c}\right)\frac{\ell}{\tau}\bar{z}\right] \quad \text{and} \quad G(\bar{t}) = \exp\left(\frac{2\ell}{c\tau}\bar{t}\right).$$

IV. SUMMARY

In Sec. II, we have presented the analytic solution (4)–(9), representing a pair of pulses, with similar or very different shapes, that propagate with invariant shapes through an atomic medium and transfer atomic population from one bound state to another via the continuum. The pulse duration and speed depend on the amplitudes, according to Eqs. (7) and (8). We have also presented a solitary wave that satisfies Eq. (16) and propagates through the atomic medium with speed c, without producing any transfer of population in the atoms or molecules. Under certain conditions similar to those for formation of ''adiabatons'' in the lambda-type three-level medium, it seems likely that both types of solitary waves may be formed in the medium. Experimental confirmation of these results would be of considerable interest, and extension of these calculations to a model with structure in the continuum is also desirable. In Sec. III, we have presented the general solution of the coupled Schrödinger and Maxwell equations for the case of one bound state and one continuum. For a pulse sent into the medium, we have obtained Eq. (25), the nonlinear Beer's law, and we have seen that the shape and velocity of an incident pulse change with depth. Also, we have related the general solution to Eq. (32), the solitary wave that propagates with invariant shape and constant velocity.

ACKNOWLEDGMENT

This research was supported by NSF Grant No. PHY-9507837.

[1] M. J. Konopnicki and J. H. Eberly, Phys. Rev. A **24**, 2567 (1981); M. J. Konopnicki, P. D. Drummond, and J. H. Eberly, Opt. Commun. **36**, 313 (1981).
[2] F. T. Hioe, J. Opt. Soc. Am. B **6**, 335 (1989); **6**, 1245 (1989).
[3] S. E. Harris, Phys. Rev. Lett. **72**, 52 (1994); S. E. Harris, J. E. Field, and A. Imamoğlu, *ibid.* **64**, 1107 (1990).
[4] S. E. Harris, Opt. Lett. **19**, 2018 (1994).
[5] F. T. Hioe and R. Grobe, Phys. Rev. Lett. **73**, 2559 (1994).
[6] R. Grobe, F. T. Hioe, and J. H. Eberly, Phys. Rev. Lett. **73**, 3183 (1994).
[7] A. Kasapi, M. Jain, G. Y. Yin, and S. E. Harris, Phys. Rev. Lett. **74**, 2447 (1995).
[8] S. Schiemann, A. Kuhn, S. Steuerwald, and K. Bergmann, Phys. Rev. Lett. **71**, 3637 (1993); J. Lawall and M. Prentiss, *ibid.* **72**, 993 (1994); L. S. Goldner *et al.*, *ibid.* **72**, 997 (1994); J. Martin, B. W. Shore, and K. Bergmann, Phys. Rev. A **54**, 1556 (1996).
[9] F. T. Hioe, Phys. Lett. A **99**, 150 (1983); J. Oreg, F. T. Hioe, and J. H. Eberly, Phys. Rev. A **29**, 690 (1984); C. E. Carroll and F. T. Hioe, J. Opt. Soc. Am. B **5**, 1335 (1988); Phys. Rev. A **42**, 1522 (1990); J. R. Kuklinski, U. Gaubatz, F. T. Hioe, and K. Bergmann, *ibid.* **40**, 6741 (1989).
[10] C. E. Carroll and F. T. Hioe, Phys. Rev. Lett. **68**, 3523 (1992); Phys. Lett. A **199**, 145 (1995).
[11] C. E. Carroll and F. T. Hioe, in *Coherence and Quantum Optics VII*, edited by J. H. Eberly, L. Mandel, and E. Wolf (Plenum, New York, 1996), p. 527; Phys. Rev. A **54**, 5147 (1996).
[12] P. L. Knight, M. A. Lauder, and B. J. Dalton, Phys. Rep. **190**, 1 (1990).
[13] L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975) [Reprint (Dover Publications, New York, 1987)].
[14] A. Rahman, R. Grobe, and J. H. Eberly, in *Coherence and Quantum Optics VII*, edited by J. H. Eberly, L. Mandel, and E. Wolf (Plenum, New York, 1996), p. 449.
[15] F. T. Hioe and C. E. Carroll, Phys. Lett. A **220**, 49 (1996).
[16] B. I. Sturman, S. G. Odoulov, and M. Yu. Goulkov, Phys. Rep. **275**, 197 (1996).
[17] A. A. Zozulya and V. T. Tikhonchuk, Kvant. Elektron. (Moscow) **15**, 1570 (1988) [Sov. J. Quantum Electron. **18**, 981 (1988)].
[18] J. Weiss, M. Tabor, and G. Carnevale, J. Math. Phys. (N.Y.) **24**, 522 (1983).
[19] A. R. Forsyth, *Theory of Differential Equations* (Cambridge University Press, Cambridge, 1906), Vol. 6, p. 143 [Reprint (Dover Publications, New York, 1959)].
[20] G. L. Brekhovskikh, V. V. Zverev, N. Y. Okladnikov, and A. I. Sokolovskaia, Opt. Commun. **53**, 59 (1985).
[21] N. V. Okladnikov, V. V. Zverev, G. L. Brekhovskikh, and A. I. Sokolovskaya, Kvant. Elektron. (Moscow) **11**, 1105 (1984) [Sov. J. Quantum Electron. **14**, 745 (1984)].
[22] S. L. McCall and E. L. Hahn, Phys. Rev. Lett. **18**, 908 (1967).