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# Lossless propagation of optical pulses through N-level systems with Gell-Mann symmetry

Foek T. Hioe
*Saint John Fisher College*, fhioe@sjfc.edu



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### Abstract

Propagation of optical pulses through atomic media consisting of atoms with N transition levels and possessing the so-called Gell-Mann symmetry is studied. An analytic solution of the appropriate Maxwell-Bloch equations having the form of simultaneous different wavelength optical solutions is presented. The special case of N = 3 was known previously.

# Lossless propagation of optical pulses through *N*-level systems with Gell-Mann symmetry

F. T. Hioe

*Department of Physics, St. John Fisher College, Rochester, New York 14618*

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Propagation of optical pulses through atomic media consisting of atoms with $N$ transition levels and possessing the so-called Gell-Mann symmetry is studied. An analytic solution of the appropriate Maxwell–Bloch equations having the form of simultaneous different wavelength optical solitons is presented. The special case of $N = 3$ was known previously.

## 1. INTRODUCTION

In this paper I present a new analytic result in the theory of coherent light propagation through an atomic medium that consists of identical atoms each with $N$ transition levels.

The phenomenon of a solitary optical pulse propagating through an atomic medium has been the subject of intensive studies for many years.[1] The phenomenon that we shall discuss involves the so-called simultons, or simultaneous (equal-velocity) optical solitons of possibly widely different wavelengths, which were first considered by Konopnicki and co-workers.[2,3] Lossless simulton propagation generally requires several conditions: The $N$ dipole-connected energy levels of each atom of the atomic medium through which the simultons propagate must have energies that are ordered in a certain way, the atomic medium must be partially excited out of its ground state in accordance with appropriate initial conditions, and the pulse amplitudes have to satisfy appropriate relations.

The result of Konopnicki *et al.* was extended previously in several directions by the author. First, a different energy-level configuration, together with a different set of conditions, was discovered, which would permit a different set of simulton propagation.[4–6] Recently an even greater stride was made when it was discovered that there were generally $N - 1$ possible conditions; each, when satisfied, would permit the atomic medium to support the propagation of specific simultons.[7,8] The latest discovery was made possible only after the analytic solution for the dynamics of an $N$-level quantum system with the so-called SU(2)-type dynamic symmetry was found.[9]

A different type of symmetry, the so-called Gell-Mann type, was introduced to problems in quantum optics by the author,[5] first for a three-level system and subsequently for a general $N$-level system.[10–12] The complete solution for the dynamics of an $N$-level system with the Gell-Mann dynamic symmetry was given by the author in Ref. 12. For an atomic medium consisting of $N$-level atoms possessing the Gell-Mann symmetry, simulton propagation is possible provided that certain conditions are satisfied. It is my aim in this paper to present the complete solution to this problem.

## 2. *N*-LEVEL SYSTEMS WITH GELL-MANN SYMMETRY

In short, an $N$-level or $N$-state quantum system whose generally time-dependent Hamiltonian can be written in or reduced to the form given by

$$\hat{H}(t) = -\hbar \begin{bmatrix} 0 & a_1a_2{}^*f(t) & 0 & a_1a_4{}^*f(t) & \cdots \\ a_2a_1{}^*f^*(t) & \delta(t) & a_2a_3{}^*f^*(t) & 0 & \\ 0 & a_3a_2{}^*f(t) & 0 & a_3a_4{}^*f(t) & \\ a_4a_1{}^*f^*(t) & 0 & a_4a_3{}^*f^*(t) & \delta(t) & \\ \cdots & & & \cdots & \end{bmatrix}, \quad (2.1)$$

where the $a$'s are arbitrary generally complex constants and the $f(t)$ and $\Delta(t)$ are arbitrary time-dependent functions, possesses what we call the Gell-Mann symmetry.[12] If the off-diagonal elements of $\hat{H}(t)$ in Eq. (2.1) are written as $a_{jk}(t)$, the special feature of the Gell-Mann symmetry for $a_{jk}(t)$ is seen to be that for $j \neq k$:

$$a_{jk}(t) = 0 \qquad \text{for } j, k \text{ with the same parity} \quad (2.2)$$

and

$$a_{jk}(t) = \begin{cases} a_ja_k{}^*f(t) & \text{for } j \text{ odd, } k \text{ even} \quad (2.3a) \\ a_ja_k{}^*f^*(t) & \text{for } j \text{ even, } k \text{ odd} \quad (2.3b) \end{cases}.$$

If the diagonal elements of $\hat{H}(t)$ in Eq. (2.1) are written as $\Delta_{nn}(t)$, then the special feature of the Gell-Mann symmetry for $\Delta_{nn}(t)$ is that

$$\Delta_{nn}(t) = \begin{cases} 0 & \text{for all odd } n \quad (2.4a) \\ \Delta(t) & \text{for all even } n \quad (2.4b) \end{cases}.$$

In the laser-driven atomic system, let level 1 be the ground state and let us label the other states or levels so that the electric dipole transition rule is obeyed. Then Eq. (2.2) is automatically satisfied. If $\hat{H}(t)$ refers to the Hamiltonian of

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the system after the rotating-wave approximation is taken, then the nonzero off-diagonal element $a_{jk}(t)$ can be associated with the half-Rabi frequency between levels $j$ and $k$, and the diagonal elements $\Delta_{nn}(t)$ can be associated with the cumulative detuning of $n-1$ successive lasers from the corresponding sum of $n-1$ level frequencies.[13] The special feature for the Gell-Mann symmetry for the off-diagonal elements, Eqs. (2.3), means that the $(N^2-1)/2$ (for odd $N$) or $N^2/2$ (for even $N$) nonzero interaction parameters $a_{jk}(t)$ cannot all be arbitrary; the number of allowed arbitrary parameters is only $2(N-1)$ if $a_{jk}(t)$ are all complex or is only $N-1$ if $a_{jk}(t)$ are all real. Notice that for $N=3$, Eqs. (2.3) do not entail any condition for the off-diagonal elements at all because $(N^2-1)/2 = 2(N-1)$ for $N=3$. The special feature for the Gell-Mann symmetry for the diagonal elements, Eqs. (2.4), means that the system is operated at two-photon resonance (for every three successive levels) and equal one-photon detunings for all times.

The remarkable properties of systems possessing the Gell-Mann symmetry follow from the fact, shown in Ref. 12, that $\hat{H}(t)$ as given by Eq. (2.1) can be unitarily transformed by a time-independent unitary matrix $\hat{U}$ into $\hat{\mathcal{H}}(t)$ according to

$$\hat{\mathcal{H}}(t) = \hat{U}^\dagger \hat{H}(t)\hat{U}, \tag{2.5}$$

where $\hat{\mathcal{H}}(t)$ is given by

$$\hat{\mathcal{H}}(t) = -\hbar \begin{bmatrix} \begin{matrix} 0 & h_{12}(t) \\ h_{21}(t) & \Delta(t) \end{matrix} & 0 & 0 & \cdots \\ 0 & \Delta(t) & 0 & \\ 0 & 0 & \Delta(t) & \\ & & & \cdots \end{bmatrix}, \tag{2.6}$$

where

$$h_{12}(t) = h_{21}^*(t) = M_1 M_2 f(t), \tag{2.7}$$

$$M_1 = (|a_1|^2 + |a_3|^2 + |a_5|^2 + \ldots)^{1/2}, \tag{2.8a}$$

$$M_2 = (|a_2|^2 + |a_4|^2 + |a_6|^2 + \ldots)^{1/2}, \tag{2.8b}$$

$$M_1 M_2 = \left( \sum_{\substack{j<k \\ (j,k)}} \sum |a_j a_k^*|^2 \right)^{1/2}, \tag{2.9}$$

and where $(j, k)$ denotes $j$ and $k$ having different (odd or even) parity. The set of constants of evolution when the system possesses the Gell-Mann symmetry has been shown to resemble closely the set of quantum numbers in elementary particle physics. Reference 12 showed how the unitary matrix $\hat{U}$, which transformed $\hat{H}(t)$ into $\hat{\mathcal{H}}(t)$, could be constructed.

The time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}(t)\psi \tag{2.10}$$

can be written as

$$i\hbar \frac{\partial \psi'}{\partial t} = \hat{\mathcal{H}}(t)\psi', \tag{2.11}$$

where

$$\psi' = \hat{U}^\dagger \psi. \tag{2.12}$$

We shall now use the density-matrix formulation, first generally, and then we will describe the special features of the Gell-Mann symmetry.

Let $\hat{s}_j$, $j = 1, 2, \ldots, N^2-1$ represent a set of traceless generators of SU($N$) algebra. It consists of three groups of $N \times N$ matrices denoted by $\hat{u}_{jk}$, $\hat{v}_{jk}$, and $\hat{w}_n$, where $N \geq k > j = 1, 2, \ldots, N-1$ and $n = 2, \ldots, N$. The elements of $\hat{u}_{jk}$ and $\hat{v}_{jk}$ are zeros except for the elements $(j, k)$ and $(k, j)$, which are equal to 1 for $\hat{u}_{jk}$ and equal to $-i$ and $i$, respectively, for $\hat{v}_{jk}$. The elements of $\hat{w}_n$ are zeros except for the elements along the diagonal, which are equal to $[2/n(n-1)]^{1/2}$ for the elements $(1, 1), (2, 2), \ldots, (n-1, n-1)$ and equal to $-[2/n(n-1)]^{1/2}(n-1)$ for the element $(n, n)$. It can be verified that the generators $\hat{s}_j$ so constructed satisfy the relation

$$\mathrm{tr}(\hat{s}_j \hat{s}_k) = 2\delta_{jk}. \tag{2.13}$$

In terms of the set $\hat{s}_j$ and the unit matrix $\hat{I}$, the density matrix $\hat{\rho}(t)$ can be written as

$$\hat{\rho}(t) = \frac{1}{N}\hat{I} + \frac{1}{2}\sum_{j=1}^{N^2-1} s_j(t)\hat{s}_j, \tag{2.14}$$

where

$$s_j(t) = \mathrm{tr}(\hat{\rho}(t)\hat{s}_j), \tag{2.15}$$

and the Hamiltonian of the system can be written as

$$\hat{H}(t) = a\hat{I} + \frac{1}{2}\sum_{j=1}^{N^2-1} \Gamma_j(t)\hat{s}_j, \tag{2.16}$$

where

$$\Gamma_j(t) = \mathrm{tr}[\hat{H}(t)\hat{s}_j]. \tag{2.17}$$

The coefficient $a$ of the unit operator $\hat{I}$ in Eq. (2.16) is not important because we may add to $\hat{H}(t)$ any multiple of a unit operator without affecting the important physical properties of the system.

Let $\hat{U}$ be a time-independent unitary matrix and $\hat{U}^\dagger$ be its complex-conjugate transpose, and let us denote

$$\hat{\rho}'(t) \equiv \hat{U}^\dagger \hat{\rho}(t)\hat{U} \tag{2.18}$$

and

$$\hat{s}_j' \equiv \hat{U}\hat{s}_j\hat{U}^\dagger. \tag{2.19}$$

Note the difference in the order of unitary transformations in Eqs. (2.18) and (2.19). Then it is easy to show that we can write

$$\hat{\rho}'(t) = \frac{1}{N}\hat{I} + \frac{1}{2}\sum_{j=1}^{N^2-1} s_j'(t)\,\hat{s}_j \tag{2.20}$$

and

$$\hat{\rho}(t) = \frac{1}{N}\hat{I} + \frac{1}{2}\sum_{j=1}^{N^2-1} s_j'(t)\,\hat{s}_j', \tag{2.21}$$

where

$$s_j'(t) = \mathrm{tr}[\hat{\rho}(t)'\hat{s}_j] = \mathrm{tr}[\hat{\rho}(t)\hat{s}_j']. \tag{2.22}$$

Similarly, if we denote

$$\hat{\mathscr{H}}(t) = \hat{U}^{\dagger}\hat{H}(t)\hat{U}, \tag{2.23}$$

it easily follows that we can write

$$\hat{\mathscr{H}}(t) = a\hat{I} + \frac{1}{2}\sum_{j=1}^{N^2-1} \Gamma_j'(t)\,\hat{s}_j \tag{2.24}$$

and

$$\hat{H}(t) = a\hat{I} + \frac{1}{2}\sum_{j=1}^{N^2-1} \Gamma_j'(t)\,\hat{s}_j', \tag{2.25}$$

where

$$\Gamma_j'(t) = \mathrm{tr}[\hat{\mathscr{H}}(t)\hat{s}_j] = \mathrm{tr}[\hat{H}(t)\hat{s}_j']. \tag{2.26}$$

The advantage of using the transformed Eqs. (2.20)–(2.26) will become apparent when $\hat{H}(t)$ possesses certain symmetries such as the Gell-Mann symmetry, as we shall describe below.

The Liouville equation governing the evolution of the density matrix $\hat{\rho}(t)$ is given by

$$i\hbar\frac{\partial\hat{\rho}(t)}{\partial t} = [\hat{H}(t), \hat{\rho}(t)]. \tag{2.27}$$

It is often more convenient to consider instead the evolution of the $(N^2 - 1)$-dimensional real coherence vector $\mathbf{S}(t) = [s_1(t), s_2(t), \ldots, s_{N^2-1}(t)]$, where the components $s_j(t)$ of this coherence vector are given in terms of the density-matrix elements $\rho_{jk}(t)$ by Eq. (2.15). In terms of the column coherence vector $\mathbf{S}(t)$, the Liouville equation can be written as

$$\frac{\mathrm{d}s_j}{\mathrm{d}t} = \sum_{k=1}^{N^2-1} A_{jk}(t)s_k(t),\ j = 1, 2, \ldots, N^2 - 1 \tag{2.28}$$

or

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} = \hat{A}(t)\mathbf{S}(t), \tag{2.29}$$

where the matrix elements $A_{jk}(t)$ of the matrix $\hat{A}(t)$ are given by

$$A_{jk}(t) = -\frac{1}{2i\hbar}\mathrm{tr}\{\hat{H}(t)[\hat{s}_j, \hat{s}_k]\}; \tag{2.30}$$

or, in terms of the transformed variables given in Eqs. (2.19), (2.22), and (2.23), we have

$$\frac{\mathrm{d}s_j'}{\mathrm{d}t} = \sum_{k=1}^{N^2-1} A_{jk}'(t)s_k'(t),\ j = 1, 2, \ldots, N^2 - 1 \tag{2.31}$$

or

$$\frac{\mathrm{d}\mathbf{S}'}{\mathrm{d}t} = \hat{A}'(t)\mathbf{S}'(t), \tag{2.32}$$

where

$$A_{jk}'(t) = -\frac{1}{2i\hbar}\mathrm{tr}\{\hat{H}(t)[\hat{s}_j', \hat{s}_k']\} = -\frac{1}{2i\hbar}\mathrm{tr}\{\hat{\mathscr{H}}(t)[\hat{s}_j, \hat{s}_k]\}. \tag{2.33}$$

We shall now show the special feature of Eq. (2.32) if the Hamiltonian $\hat{H}(t)$ of the system possesses the Gell-Mann symmetry, as given in Eq. (2.1), so that a time-independent unitary matrix $\hat{U}$ can be found that could transform $\hat{H}(t)$ into $\mathscr{H}(t)$, as given by Eq. (2.6). Just as we have used $s_j(t)$ to denote the expectation value of $\hat{s}_j$ as given by Eq. (2.15), we shall use $u_{jk}(t)$, $v_{jk}(t)$, and $w_n(t)$ to denote the expectation values of $\hat{u}_{jk}(t)$, $\hat{v}_{jk}(t)$, and $\hat{w}_n(t)$, respectively, where the corresponding operators were defined following Eq. (2.12). Similarly, we shall use $u_{jk}'(t)$, $v_{jk}'(t)$, and $w_n'(t)$ to denote the expectation values of $\hat{u}_{jk}'$, $\hat{v}_{jk}'$, and $\hat{w}_n'$, respectively, or of $\hat{u}_{jk}$, $\hat{v}_{jk}$, and $\hat{w}_n$, respectively, according to Eq. (2.22). The transformed coherence vector is $\mathbf{S}'(t) = [s_1'(t), s_2'(t), \ldots, s_{N^2-1}'(t)]$, where the components $s_j'(t)$ are given by Eq. (2.22). Using Eqs. (2.33) and (2.6), we can easily show that, as a consequence of the Gell-Mann symmetry, the $(N^2 - 1)$-dimensional space in which the coherence vector $\mathbf{S}'(t)$ evolves can be decomposed into the following subspaces in which the components of $\mathbf{S}'(t)$ evolve independently of one another: one three-dimensional, $N - 2$ four-dimensional, $(N - 2)(N - 3)/2$ two-dimensional, and $N - 2$ one-dimensional subspaces. Let us denote the following column (col) vectors:

$$\mathbf{S}_3' = \mathrm{col}(u_{12}', v_{12}', w_2'),$$

$$\mathbf{S}_{4,n}' = \begin{cases} \mathrm{col}(u_{1n}', v_{1n}', u_{2n}', v_{2n}') & \text{for } n \text{ odd} \\ \mathrm{col}(u_{2n}', -v_{2n}', -u_{1n}', v_{1n}') & \text{for } n \text{ even} \end{cases},$$

$$\mathbf{S}_{2D,mn}' = \begin{cases} \mathrm{col}(-u_{mn}', v_{mn}') & \text{for } m \text{ odd}, n \text{ even} \\ \mathrm{col}(u_{mn}', v_{mn}') & \text{for } m \text{ even}, n \text{ odd} \end{cases},$$

$$\mathbf{S}_{2S,mn}' = \begin{cases} \mathrm{col}(u_{mn}', v_{mn}') & \text{for } m \text{ odd}, n \text{ odd} \\ \mathrm{col}(u_{mn}', -v_{mn}') & \text{for } m \text{ even}, n \text{ even} \end{cases},$$

$$\mathbf{S}_{1,n}' = \mathrm{col}(w_n'), \tag{2.34}$$

where the first subscripts for $\mathbf{S}$ refer to the dimensions of the subspaces, the second subscripts after the commas refer to the particular vectors with $m, n = 3, 4, \ldots, N$, and with $n > m$, and the subscripts $D$ and $S$ for $\mathbf{S}_2'$ refer to $m$, $n$ having different and same parity, respectively. The $N^2 - 1$ components of the coherence vector $\mathbf{S}'$ consist of one $\mathbf{S}_3'$, $(N - 2)\mathbf{S}_4'$, $(N - 2)(N - 3)/2$ $\mathbf{S}_2'$, and $N - 2$ $\mathbf{S}_1'$. The equations of motion for these vectors can be verified to be independent of one another, and they are given explicitly in Appendix A. In particular, the equations of motion for $\mathbf{S}_3'(t)$ are of the form of the Bloch equations for the effective two-level system, and they are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{S}_3'(t) = \hat{A}_3'(t)\mathbf{S}_3'(t), \tag{2.35}$$

where

$$\hat{A}_3'(t) = \begin{bmatrix} 0 & -\Delta(t) & \Omega_i(t) \\ \Delta(t) & 0 & \Omega_r(t) \\ -\Omega_i(t) & -\Omega_r(t) & 0 \end{bmatrix}. \tag{2.36}$$

Here the matrix elements in $\hat{A}_3'(t)$ are related to the matrix elements of $\hat{\mathscr{H}}(t)$ in Eq. (2.6) by

$$h_{12}(t) \equiv \tfrac{1}{2}\Omega(t) = \tfrac{1}{2}\{\Omega_r(t) + i\Omega_i(t)\}. \tag{2.37}$$

We have adopted the notations used in quantum optics in Eq. (2.37). Similarly in Eq. (2.1), we shall denote

$$\tfrac{1}{2}\Omega_{jk}(t) \equiv a_j a_k^* f(t) \qquad \text{for } j \text{ odd, } k \text{ even,} \tag{2.38}$$

and refer to $\Omega_{jk}(t)$ as the Rabi frequency between levels $j$ and $k$. In this terminology $\Omega_r(t)$ and $\Omega_i(t)$ in Eqs. (2.36) and (2.37) are the real and imaginary parts, respectively, of the effective Rabi frequency $\Omega(t)$, which, according to Eqs. (2.7)–(2.9), is given in terms of the individual Rabi frequencies $\Omega_{jk}(t)$ by

$$\Omega(t) = \left(\sum_{j<k}\sum |\Omega_{jk}(t)|^2\right)^{1/2} = 2\left(\sum_{j<k}\sum |a_j a_k^*|^2\right)^{1/2} f(t). \tag{2.39}$$

The summations in Eq. (2.39) are taken over $j, k = 1, 2, \ldots, N$, with $j, k$ having different parity [for which $\Omega_{jk}(t) \neq 0$].

The grouping of the components of the coherence vector $\mathbf{S}'$ into subgroups shown in Eqs. (2.34) is characteristic of the Gell-Mann symmetry reminiscent of the grouping of the pseudoscalar mesons in quark physics. For example, for $N = 3$ (3 quark flavors) the components of $\mathbf{S}'$ (the mesons) divide into a group of three (the three pions), a group of four (the four kaons), and a group of one (the eon). In a future publication[14] we shall show the groupings for a general value of $N$ according to the equations of motion of systems possessing the Gell-Mann symmetry, and shall show how they resemble the groupings of particles associated with pseudoscalar mesons as well as with those associated with vector mesons and baryons.

## 3. SIMULTON PROPAGATION

We assume a plane-wave incident electric field $\mathbf{E}(z, t)$ propagating in the $z$ direction with a number of frequency components:

$$\mathbf{E}(z, t) = \sum_{j,k} (\mathbf{e}_{jk}\mathcal{E}_{jk}(z, t)\exp\{i\nu_{jk}C_{jk}[t - (z)/c]\} + \text{c.c.}), \tag{3.1}$$

where $\nu_{jk}$ denotes the (circular) carrier frequency of the component nearly at resonance with the transition frequency between levels $j$ and $k$, $\mathbf{e}_{jk}$ is its possibly complex polarization vector, $\mathcal{E}_{jk}(z, t)$ is its complex amplitude, assumed to be a slowly varying function of $z$ and $t$ compared to the optical frequency, $C_{jk}$ depends on the energy-level ordering so that for increasing energies $E_k > E_j$, $C_{jk} = 1$, and for decreasing energies $E_k < E_j$, $C_{jk} = -1$. The use of $C_{jk}$ permits the Bloch equations or the density-matrix equations for the evolution of the atomic variables in the rotating-wave approximation to have an invariant form for any energy-level ordering.

The evolution of the atomic system is described by the Liouville equation [Eq. (2.27)] for the density matrix $\hat{\rho}(t)$, and the propagation of the pulse envelopes is described by the reduced Maxwell equations[2,3,15]

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial(ct)}\right)\Omega_{jk}(z, t) = -i\,\frac{4\pi D}{\hbar c}\,C_{jk}\nu_{jk}d_{jk}^{\,2}\langle\rho_{jk}\rangle, \tag{3.2}$$

where $j, k$ have different parity and

$$d_{jk} = |\langle j|\mathbf{d}|k\rangle \cdot \mathbf{e}_{jk}| \tag{3.3}$$

is the appropriate component of the dipole-matrix element, in terms of which the Rabi frequency $\Omega_{jk}$ is given by

$$\Omega_{jk}(z, t) = 2(\langle j|\mathbf{d}|k\rangle \cdot \mathbf{e}_{jk})\hbar^{-1}\mathcal{E}_{jk}(z, t), \tag{3.4}$$

and where $D$ is the atomic density and the angle brackets denote averaging over the Maxwellian velocity distribution of atoms.

We shall now consider systems possessing the Gell-Mann symmetry and consider the case when all the Rabi frequencies $\Omega_{jk}$ are real. From Eqs. (2.34)–(2.36), the equation of motion for $\mathbf{S}_3'$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u_{12}' \\ v_{12}' \\ w_2' \end{bmatrix} = \begin{bmatrix} 0 & -\Delta(t) & 0 \\ \Delta(t) & 0 & \Omega(t) \\ 0 & -\Omega(t) & 0 \end{bmatrix} \begin{bmatrix} u_{12}' \\ v_{12}' \\ w_2' \end{bmatrix}. \tag{3.5}$$

Equation (3.5) is mathematically identical to the Bloch equation for the two-level system. The $\Omega(t)$ here is the effective Rabi frequency given by Eq. (2.39), and the $\Delta(t)$ here is the one-photon detuning of every carrier frequency from the two transition levels [see Eqs. (2.4)] while the system is assumed to be at two-photon resonance for any three successive levels. If we assume or impose the conditions that

$$\mathbf{S}_{4,n}'(0) = 0;\ n = 3, 4, \ldots, N \tag{3.6}$$

and

$$\mathbf{S}_{2,mn}'(0) = 0;\ m, n = 3, 4, \ldots, N;\ n > m, \tag{3.7}$$

then the solution for all components of the atomic variables is given in terms of the solutions of $u_{12}'(t)$, $v_{12}'(t)$, and $w_2'(t)$ given by Eq. (3.5) for the given initial values of $u_{12}'(0)$, $v_{12}'(0)$, and $w_2'(0)$ because, from Eqs. (3.6) and (3.7), we have

$$S_{4,n}'(t) = 0, \tag{3.8}$$

$$S_{2,mn}'(t) = 0, \tag{3.9}$$

$$S_{1,n}'(t) = S_{1,n}'(0). \tag{3.10}$$

Let us assume that initially

$$\rho_{ll'}(0) = 0 \qquad \text{for } l \neq l', \tag{3.11}$$

i.e., all the off-diagonal elements of $\hat{\rho}$ are initially equal to zero. We want to find what initial values of the diagonal elements satisfy $\rho_{ll}(0)$ so that conditions (3.6) and (3.7) can be fulfilled, i.e., we want to find the condition that would make

$$u_{jk}'(t) = 0, \tag{3.12}$$

$$v_{jk}'(t) = 0 \tag{3.13}$$

for all $j, k$ except for $j = 1, k = 2$. As we shall show in Appendix B, Eqs. (3.11)–(3.13) imply the following initial condition:

$$\rho_{ll}(0) = \begin{cases} c_1 & \text{for all odd } l \\ c_2 & \text{for all even } l \end{cases}, \tag{3.14}$$

where $c_1$ and $c_2$ are arbitrary constants that satisfy

$$\sum_l \rho_{ll}(0) = 1. \tag{3.15}$$

Equation (3.14) reduces the atomic evolution problem to the following: The only equation that we need to solve is Eq. (3.5) for $u_{12}'(t) \equiv \rho_{12}'(t) + \rho_{21}'(t)$, $v_{12}'(t) \equiv i(\rho_{12}'(t) - \rho_{21}'(t)$, and $w_2'(t) \equiv \rho_{11}'(t) - \rho_{22}'(t)$, with the initial condition that $u_{12}'(0) = v_{12}'(0) = 0$ and $w_2'(0) \equiv \rho_{11}'(0) - \rho_{22}'(0) = \rho_{11}(0) - \rho_{22}(0) = c_1 - c_2$. From these solutions, we can find the solutions for all $\rho_{jk}(t)$ as follows (see Appendix B):

$$\rho_{ml}(t) = \begin{cases} \rho_{12}'(t) r_m^{(1)} r_l^{(2)*} & \text{for } m \text{ odd, } l \text{ even} \\ \rho_{21}'(t) r_m^{(2)} r_l^{(1)*} & \text{for } m \text{ even, } l \text{ odd} \end{cases}, \tag{3.16}$$

$$\rho_{jk}(t) = \sum_l \rho_{ll}'(t) r_j^{(l)} r_k^{(l)*} \qquad \text{for } j, k \text{ with the same parity,} \tag{3.17}$$

where

$$\rho_{ll}'(t) = \begin{cases} c_1 & \text{for all odd } l \\ c_2 & \text{for all even } l \end{cases}, \tag{3.18}$$

$$\rho_{11}'(t) + \rho_{22}'(t) = c_1 + c_2, \tag{3.19}$$

and $r_j^{(k)}$ is the $(j, k)$th matrix elements of the unitary matrix $\hat{U}$ given in Ref. 12. In particular, we have

$$r_j^{(k)} = \begin{cases} M_k^{-1} a_j & \text{for } k = 1 \text{ or } 2 \text{ and } j, k \text{ of the same parity} \\ 0 & \text{for } j, k \text{ of different parity} \end{cases}, \tag{3.20}$$

where $M_1$ and $M_2$ are given by Eq. (2.8).

We shall now check whether the solution for the atomic evolution so obtained can be used in a consistent manner in conjunction with the Maxwell Eq. (3.2). From Eqs. (3.2), where $j$, $k$ have different parity, (3.16), (3.20), (2.3), and (2.38), we can write, for $\Omega_{jk}$, which are assumed real, and for $j$ odd, $k$ even, that

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial(ct)}\right) 2a_j a_k f(z, t) = -i \frac{2\pi D}{\hbar c} C_{jk} \nu_{jk} d_{jk}^2 a_j a_k [\rho_{12}'(z, t) - \rho_{21}'(z, t)] \tag{3.21a}$$

and

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial(ct)}\right) 2a_k a_j f(z, t) = -i \frac{2\pi D}{\hbar c} C_{kj} \nu_{kj} d_{kj}^2 a_k a_j [\rho_{21}'(z, t) - \rho_{12}'(z, t)]. \tag{3.21b}$$

Canceling the common factor $a_j a_k$ from both sides of Eqs. (3.21), Eqs. (3.21a) and (3.21b) can be made to be consistent if the following conditions are met:

(i) All the odd-numbered energy levels have energies lower (or higher) than all the even-numbered energy levels [see the definition of $C_{jk}$ below Eq. (3.1)].

(ii) $\nu_{jk} d_{jk}^2$ = the same constant for all $j$, $k$ with different parity.

The quantities in condition (ii) are proportional to the oscillator strengths[16] of the respective transitions, and condition (ii) implies that the oscillator strengths of all allowed transitions must be equal. Assuming that conditions (i) and (ii) above are satisfied, then the set of Maxwell equations for all the incident pulses reduces to a single equation given by, for $j$ odd, $k$ even,

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial(ct)}\right) f(z, t) = -\frac{\pi D}{\hbar c} C_{jk} d_{jk}^2 \nu_{jk} \langle v_{12}'(z, t)\rangle, \tag{3.22}$$

where $f(t)$ is the common time-dependent envelope of the incident pulses, or

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial(ct)}\right) \Omega(z, t) = -\frac{2\pi D}{\hbar c} C_{jk} d_{jk}^2 \nu_{jk} M_1 M_2 \langle v_{12}'(z, t)\rangle, \tag{3.23}$$

where $\Omega(t)$ is the effective Rabi frequency given by Eq. (2.39) and $M_1 M_2$ is given by Eq. (2.9).

Let

$$\zeta \equiv t - \frac{z}{V}, \tag{3.24}$$

where $V$ denotes the common velocity of the pulses, so that

$$\frac{\partial}{\partial t} \to \frac{d}{d\zeta}, \qquad \frac{\partial}{\partial z} \to -\frac{1}{V}\frac{d}{d\zeta}, \tag{3.25}$$

and let

$$\Theta(z, t) \equiv \int_{-\infty}^{t} \Omega(z, t')dt'. \tag{3.26}$$

Then Eq. (3.23) can be written, for $j$ odd, $k$ even, as

$$\ddot{\Theta} = \frac{2\pi D}{\hbar c\left(\frac{1}{V} - \frac{1}{c}\right)} C_{jk} \nu_{jk} d_{jk}^2 M_1 M_2 \langle v_{12}'\rangle, \tag{3.27}$$

where the dot denotes the derivative with respect to $\zeta$. Equation (3.27), solved in conjunction with Eq. (3.5) (for $\Delta \simeq 0$), gives

$$\ddot{\Theta} = \frac{1}{\tau^2} \sin\Theta, \tag{3.28}$$

where the pulse length $\tau$ is related to the velocity $V$ of the simultons by

$$\frac{1}{\tau^2} = \frac{2\pi D}{\hbar c\left(\frac{1}{V} - \frac{1}{c}\right)} M_1 M_2 C_{jk}(c_1 - c_2) \nu_{jk} d_{jk}^2, \tag{3.29}$$

$c_1$, $c_2$ being given by Eq. (3.14). The requirement that $1/\tau^2$ be positive for Eq. (3.28) in order to give a soliton solution means that in addition to Eqs. (3.11) and (3.14) and conditions (i) and (ii), we need one more condition that

$$C_{jk}(c_1 - c_2) > 0 \quad \text{or} \quad C_{kj}(c_1 - c_2) < 0 \tag{3.30}$$

for $j$ odd, $k$ even. The solution for $\Theta$ is

$$\Theta(z, t) = 4\tan^{-1}\left[\exp\left(\frac{\zeta - \zeta_0}{\tau}\right)\right], \tag{3.31}$$

and the associated solution for $\Omega(z, t)$ is[15]

$$\Omega(z, t) = \frac{2}{\tau}\operatorname{sech}\left(\frac{\zeta - \zeta_0}{\tau}\right). \tag{3.32}$$

The electric-field envelopes of the set of simultons are given by all $(N^2 - 1)/4$ (for $N$ odd) or $N^2/4$ (for $N$ even) real $\Omega_{jk}(z, t)$ for which $j$, $k$ have different parity, which are given in terms of the effective Rabi frequency $\Omega(z, t)$ of Eq. (3.32) by

$$\Omega_{jk}(z, t) = M_1^{-1}M_2^{-1}a_ja_k\Omega(z, t), \qquad (3.33)$$

where the parameters are defined in Eqs. (2.3), (2.9), and (2.38).

The corresponding area theorem of McCall and Hahn[17] is

$$\frac{\partial}{\partial z}\Theta(z, t) = -\frac{1}{2}\alpha \sin\Theta, \qquad (3.34)$$

where $\alpha$ is the absorption coefficient. Thus, if $\Theta = n\pi$ for any positive integer $n$, the pulse-envelope area suffers no attenuation in propagation since $\partial\Theta/\partial z = 0$. Thus, given Eqs. (2.1), (3.11), (3.14), and (3.30) and conditions (i) and (ii), the set of simultons, whose electric-field envelopes are given by Eqs. (3.32) and (3.33) and which may have different wavelengths determined by condition (ii), can propagate through the medium with the same velocity $V$ determined from their common pulse length $\tau$ by Eq. (3.29) without attenuation. The pulses behave as if the medium were transparent. The time difference between $V$ and $c$ in traveling through a length $L$ of the atomic medium is given by

$$\tau_d = \frac{L}{V} - \frac{L}{c} = \frac{1}{2}(\alpha L). \qquad (3.35)$$

Our simulton solution, Eqs. (3.32) and (3.33), reduces to a three-level case discovered by Konopnicki and co-workers[2,3] for $N = 3$, but for the general $N > 3$ case our solution and the conditions that would make the simulton propagation possible are new. The conditions and the solution are quite different from those corresponding to systems possessing the SU(2) symmetry.[8]

The combination of the conditions specified by Eqs. (3.30) and (3.18) means the following: As we number the $N$ energy levels of the atom of the medium by $1, 2, \ldots, N$ so that the electric dipole moments connect only energy levels of different parity, the odd-numbered energy levels are required to have energies that are (a) all lower or (b) all higher than the even-numbered energy levels. The atomic medium must be in or prepared in a state in which the initial level populations of all the odd-numbered levels are equal (to $c_1$) and the initial-level populations of all the even-numbered levels are equal (to $c_2$), where $c_1$ and $c_2$ are restricted by the following inequalities:

$$1/N < c_1 \le 1/n_o, \qquad 0 \le c_2 < 1/N \qquad (3.36a)$$

for condition (a) or

$$0 \le c_1 < 1/N, \qquad 1/N < c_2 \le 1/n_o \qquad (3.36b)$$

for condition (b).

Here $n_o$ denotes the number of odd-numbered energy levels. Note that $n_o$ is equal to $n_e$, the number of even-numbered energy levels if the total number of energy levels $N$ is even, and that $n_o$ is equal to $n_e + 1$ if $N$ is odd. A permissible population distribution for condition (a), for example, is

$$\rho_{ll}(0) = \begin{cases} 1/n_o & \text{for } l \text{ odd} \\ 0 & \text{for } l \text{ even} \end{cases}. \qquad (3.37)$$

## 4. SOME EXAMPLES

In this section we shall illustrate the conditions for simulton propagation with some specific examples.

### Case (I) $N = 3$

(A) The three relevant energy levels of each atom of the atomic medium have a $\Lambda$-type configuration shown in Fig. 1.

(B) The two incident laser pulses characterized by the Rabi frequencies $\Omega_{12}$ and $\Omega_{23}$ given by Eq. (3.4) may have any arbitrary amplitudes but have the same common time- and space-dependent factor given by Eq. (3.32).

(C) The frequencies of the incident laser electric fields and the electric dipole moments connecting the levels are required to satisfy the relation

$$\nu_{12}d_{12}^2 = \nu_{23}d_{23}^2.$$

(D) The initial-level populations are arranged to be given by

$$\rho_{11}(0) = \rho_{33}(0) = c_1,$$

$$\rho_{22}(0) = c_2,$$

where $1/3 < c_1 \le 1/2$ and $0 \le c_2 < 1/3$. Two specific examples are (a) $c_1 = 1/2$, $c_2 = 0$ and (b) $c_1 = 4/9$, $c_2 = 1/9$.

### Case (II) $N = 4$

(A) The four relevant energy levels of the atom have a configuration shown in Fig. 2.

(B) Among the four Rabi frequencies $\Omega_{12}$, $\Omega_{23}$, $\Omega_{34}$, and $\Omega_{14}$ of the four incident laser pulses, three may have any arbitrary amplitudes, but the fourth must satisfy the relation

$$\Omega_{12}\Omega_{34} = \Omega_{14}\Omega_{23}.$$

(C) The frequencies of the incident electric fields and the electric dipole moments connecting the levels are required to satisfy the relation

$$\nu_{12}d_{12}^2 = \nu_{23}d_{23}^2 = \nu_{34}d_{34}^2 = \nu_{14}d_{14}^2.$$

(D) The initial-level populations are arranged so that

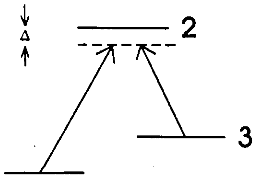


Fig. 1. Schematic representation of a three-level system interacting with the two laser pulses described in Section 4.

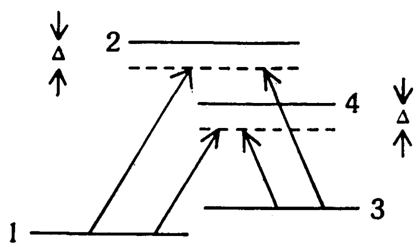


Fig. 2. Schematic representation of a four-level system interacting with the four laser pulses described in Section 4.

$$\rho_{11}(0) = \rho_{33}(0) = c_1,$$
$$\rho_{22}(0) = \rho_{44}(0) = c_2,$$

where $1/4 < c_1 \le 1/2$ and $0 \le c_2 < 1/4$. Two specific examples are (a) $c_1 = 1/2$, $c_2 = 0$ and (b) $c_1 = 4/9$, $c_2 = 1/18$.

### Case (III) General *N*

(A) The $N$ energy levels of each atom of the atomic medium are assumed to fall into two bands: one band consisting of all odd-numbered levels has approximately the same energy $E_1$, and the other band consisting of all even-numbered levels has approximately the same energy $E_2$, and $E_2 > E_1$.

(B) and (C) The $(N^2 - 1)/4$ (for $N$ odd) or $N^2/4$ (for $N$ even) Rabi frequencies $\Omega_{jk}$, electric dipole moments $d_{jk}$, and frequencies $\nu_{jk}$ of the laser pulses for $j$, $k$ of different parity satisfy the following relations:

All $\Omega_{jk}$ are approximately equal;
All $\nu_{jk}d_{jk}^2$ are approximately equal.

(D) The initial-level populations are given by inequalities (3.36a).

When the above conditions are satisfied, the systems described in cases (I), (II), and (III) above would permit simulton or approximate simulton [for case (III)] propagation. If one or more of the prescribed conditions are not satisfied in practice, then the prescribed conditions are still useful in the sense that they can be viewed as a set of ideal conditions from which the experimental conditions may have differed.

## 5. SUMMARY

We began with an atomic medium in which each atom generally has $N$ transition levels, and we considered sending $(N^2 - 1)/4$ (for $N$ odd) or $N^2/4$ (for $N$ even) simultaneous equal-velocity laser pulses through the medium.

We assumed that the time-dependent Hamiltonian of the laser-atom interacting system could be written in or reduced to the form given by Eq. (2.1). The system was said to possess the Gell-Mann symmetry. The physical conditions required for the system to exhibit the Gell-Mann symmetry were explained following Eqs. (2.4).

We assumed that conditions expressed by Eqs. (3.11), (3.14), and (3.30) and conditions (i) and (ii) [given below Eqs. (3.21)] were also satisfied.

We showed that when the above conditions were satisfied, then the set of $(N^2 - 1)/4$ (for $N$ odd) or $N^2/4$ (for $N$ even) laser pulses given by Eqs. (3.32) and (3.33) constitutes simultons that can propagate through the atomic medium without attenuation.

## APPENDIX A

In this appendix we shall write the equations of motion for the transformed coherence vector $\mathbf{S}'(t)$ when the system possesses the Gell-Mann symmetry.

With the components of the vectors arranged as in Eqs. (2.34), if we write $f$ for the first subscript and $s$ for the second subscript (when there is one) for these coherent vectors, the transition matrices for the time evolution of $\mathbf{S}_{f,s}'$ are independent of the second subscript $s$, and the equations of motion can be written generally as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{S}_{f,s}'(t) = \hat{A}_f'(t)\mathbf{S}_{f,s}', \tag{A1}$$

where $\hat{A}_3'(t)$ is given by Eq. (2.36) and

$$\hat{A}_4'(t) = \begin{bmatrix} 0 & 0 & -\tfrac{1}{2}\Omega_i(t) & \tfrac{1}{2}\Omega_r(t) \\ 0 & 0 & -\tfrac{1}{2}\Omega_r(t) & -\tfrac{1}{2}\Omega_i(t) \\ \tfrac{1}{2}\Omega_i(t) & \tfrac{1}{2}\Omega_r(t) & 0 & \Delta(t) \\ -\tfrac{1}{2}\Omega_r(t) & \tfrac{1}{2}\Omega_i(t) & -\Delta(t) & 0 \end{bmatrix},$$
$$\hat{A}_{2D}'(t) = \begin{bmatrix} 0 & \Delta(t) \\ -\Delta(t) & 0 \end{bmatrix}, \qquad \hat{A}_{2S}'(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\hat{A}_1'(t) = [0]. \tag{A2}$$

## APPENDIX B

In this appendix we show that the imposition of the conditions given by Eqs. (3.6) and (3.7) implies the initial conditions given by Eq. (3.18) and that, as a result of these initial conditions, Eqs. (3.16) and (3.17) follow.

The matrix elements $r_{jk}$ of the time-independent unitary matrix $\hat{U}$, which transforms $\hat{H}(t)$ of Eq. (2.1) into $\hat{\mathscr{H}}(t)$ of Eq. (2.6), were shown to have the form[12]

$$r_{jk} = \begin{cases} r_j^{(k)} & \text{for } j, k \text{ having same parity} \\ 0 & \text{for } j, k \text{ having different parity} \end{cases}. \tag{B1}$$

Using Eqs. (2.22) and (3.11), we find that when $j$ and $k$ have the same parity,

$$u_{jk}'(0) = \mathrm{tr}[\hat{\rho}(0)\hat{U}\hat{u}_{jk}\hat{U}^\dagger] = 2\,\mathrm{Re}\sum_l \rho_{ll}(0)r_l^{(j)}r_l^{(k)*}, \tag{B2}$$

$$v_{jk}'(0) = \mathrm{tr}[\hat{\rho}(0)\hat{U}\hat{v}_{jk}\hat{U}^\dagger] = 2\,\mathrm{Im}\sum_l \rho_{ll}(0)r_l^{(j)}r_l^{(k)*}, \tag{B3}$$

where the summation is taken over all values of $l$ that have the same parity as $j$ and $k$. From the relation

$$\hat{U}^\dagger\hat{U} = \mathrm{I} \tag{B4}$$

or

$$\sum_l r_l^{(j)}r_l^{(k)*} = \delta_{jk} \tag{B5}$$

it can be seen from Eqs. (B2) and (B3) that if we set

$$\rho_{ll}(0) = \begin{cases} c_1 & \text{for all odd } l \\ c_2 & \text{for all even } l \end{cases}, \tag{B6}$$

where

$$\sum_l \rho_{ll}(0) = 1,$$

then for all $j$, $k$ with the same parity, $u_{jk}'(0) = v_{jk}'(0) = 0$, and hence, from Eq. (2.35),

$$u_{jk}'(t) = v_{jk}'(t) = 0 \qquad \text{for all } j, k \text{ with the same parity.} \tag{B7}$$

From Eq. (2.18) and the fact that all $\rho_{ll'}(t) = 0$ for all $l \neq l'$ with the same parity [see Eq. (B7)], we find that

$$\rho_{jk}(t) = \sum_{l} r_j^{(l)} r_k^{(l)*} \rho_{ll}'(t). \tag{B8}$$

From Eq. (2.18), we also find that

$$\rho_{jj}'(0) = \sum_{l} r_j^{(l)*} r_j^{(l)} \rho_{ll}(0). \tag{B9}$$

Using Eqs. (B1) and (B6), we get

$$\rho_{ll}'(0) = \begin{cases} c_1 & \text{for all odd } l \\ c_2 & \text{for all even } l \end{cases}. \tag{B10}$$

Equation (3.10) further implies that for $l = 3, 4, \ldots, N$,

$$\rho_{ll}'(t) = \begin{cases} c_1 & \text{for all odd } l \\ c_2 & \text{for all even } l \end{cases}, \qquad l \neq 1 \text{ or } 2 \tag{B11}$$

for all $t$, and for $l = 1$ and 2 we have

$$\rho_{11}'(t) + \rho_{22}'(t) = c_1 + c_2. \tag{B12}$$

When $j$ and $k$ have different parity,

$$u_{jk}'(t) = \sum_{\substack{m \\ (k)}} \sum_{\substack{l \\ (j)}} \rho_{ml}(t) r_l^{(j)} r_m^{(k)*} + \sum_{\substack{m \\ (j)}} \sum_{\substack{l \\ (k)}} \rho_{ml}(t) r_l^{(k)} r_m^{(j)*}, \tag{B13}$$

$$v_{jk}'(t) = -i \left\{ \sum_{\substack{m \\ (k)}} \sum_{\substack{l \\ (j)}} \rho_{ml}(t) r_l^{(j)} r_m^{(k)*} - \sum_{\substack{m \\ (j)}} \sum_{\substack{l \\ (k)}} \rho_{ml}(t) r_l^{(k)} r_m^{(j)*} \right\}, \tag{B14}$$

where

$$\sum_{\substack{m \\ (k)}}$$

means that it is taken over all values of $m$ that have the same parity as $k$. If we set

$$\rho_{ml}(t) = \begin{cases} \rho(t) r_m^{(1)} r_l^{(2)*} & \text{for } m \text{ odd, } l \text{ even} \\ \rho^*(t) r_m^{(2)} r_l^{(1)*} & \text{for } m \text{ even, } l \text{ odd} \end{cases}, \tag{B15}$$

then substitutions of Eq. (B15) into Eqs. (B13) and (B14) and the use of Eq. (B5) show that for $j$, $k$ with different parity other than the values $j = 1$ and $k = 2$,

$$u_{jk}'(t) = v_{jk}'(t) = 0 \qquad \text{excluding } (j, k) = (1, 2). \tag{B16}$$

For $j = 1$ and $k = 2$, we find, from Eqs. (B13)–(B15),

$$\begin{aligned} u_{12}'(t) &= \rho_{12}'(t) + \rho_{21}'(t) \\ &= \rho(t) + \rho^*(t) \\ &= \rho_{12}(t)/r_1^{(1)} r_2^{(2)*} + \rho_{21}(t)/r_1^{(1)*} r_2^{(2)}, \end{aligned} \tag{B17}$$

$$\begin{aligned} v_{12}'(t) &= i[\rho_{12}'(t) - \rho_{21}'(t)] \\ &= i[\rho(t) - \rho^*(t)] \\ &= i[\rho_{12}(t)/r_1^{(1)} r_2^{(2)*} - \rho_{21}(t)/r_1^{(1)*} r_2^{(2)}]. \end{aligned} \tag{B18}$$

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The author is also with the Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627.

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