3-1989

Lossless propagation of optical pulses through N-level systems with SU(2) symmetry

Foek T. Hioe
Saint John Fisher College, fhioe@sjfc.edu

Publication Information

Hioe, Foek T. (1989). "Lossless propagation of optical pulses through N-level systems with SU(2) symmetry." *Journal of the Optical Society of America B* 6.3, 335-343.

Lossless propagation of optical pulses through N-level systems with SU(2) symmetry

F. T. Hioe

Department of Physics, St. John Fisher College, Rochester, New York 14618

Received August 23, 1988; accepted November 30, 1988

Propagation of optical pulses through atomic media consisting of atoms with N transition levels and possessing the so-called SU(2) symmetry is studied. It is shown that there are generally $N - 1$ sets of conditions, each of which, when satisfied, would permit the appropriate Maxwell–Bloch equations to have a solution having the form of simultaneous different-wavelength optical solitons, so-called simultons. The first two sets of solutions were known previously, but the remaining sets are new.

The concept of lossless propagation of two or more simultaneous (equal-velocity) optical solitons (simultons) that may have widely different wavelengths was introduced by Konopnicki *et al.*[1,2] A simulton, because of its multiwavelength capability, is different from pulse trains in two-level absorbers[3] and from two-photon self-induced transparency pulses.[4] Simulton propagation generally requires several conditions: the N dipole-connected energy levels (in which the dipoles connect levels n and $n + 1$ for $n = 1, 2, \ldots, N - 1$) of the atomic medium through which the $N - 1$ pulses of the simultons propagate must have energies that are ordered in a certain way; the absorbing medium must be partially excited out of its ground state in accordance with appropriate initial conditions; the pulse amplitudes have to satisfy appropriate relations.

Following the work of Konopnicki *et al.*, we extended their result by allowing nonzero equal one-photon detunings in the $N - 1$ allowed transitions[5] and also discovered that a different set of conditions on the energy-level configuration, initial-level population distribution, and strengths of the dipole matrix elements would enable an N-level system to support the propagation of a different set of simultons.[6] The two different sets of conditions that permitted the simulton propagation through an N-level system were later observed to be mutually exclusive and orthogonal.[7] The question was raised but not answered in Ref. 7 of whether other sets of conditions exist that would permit simulton propagation. For a system of N-level atoms possessing the so-called SU(2) dynamic symmetry,[8–10] the problem has now been solved in its entirety, and it is the purpose of this paper to report this solution.

The principal result that I shall present is that there are $N - 1$ sets of mutually exclusive or orthogonal conditions and that, if any one of these conditions is satisfied, it would allow the atomic medium consisting of these N-level atoms to support the propagation of simultons. The $N - 1$ sets of conditions will be labeled $k_1 = 1, 2, \ldots, N - 1$, and the precise conditions will be expressed, in a compact form, in terms of the Wigner 3-j symbols.[11] The two different sets of conditions discovered previously by Konopnicki *et al.*[1,2] and by me[6,7] will be seen to correspond to the sets characterized by $k_1 = 1$ and $k_1 = 2$, respectively.

We assume a plane-wave incident electric field $\mathbf{E}(z, t)$ with $N - 1$ distinct frequency components:

$$\mathbf{E}(z, t) = \sum_{n=1}^{N-1} \{\mathbf{e}_n \mathscr{E}_n(z, t) \exp[i\nu_n C_n(t - z/c)] + \text{c.c.}\}, \tag{1}$$

where ν_n denotes the (circular) carrier frequency of the nth component, $\mathbf{e}_n$ is its possible complex polarization vector, and $\mathscr{E}_n(z, t)$ is its complex amplitude, assumed to be a slowly varying function of z and t compared to the optical frequency. The frequencies ν_n are chosen to be nearly resonant with the successive transition frequencies in a chain of N dipole-connected energy levels in an atomic system, and C_n depends on the energy-level ordering, so that for increasing energies $E_{n+1} > E_n$, $C_n = 1$, and for decreasing energies $E_{n+1} < E_n$, $C_n = -1$. The use of C_n allows the Bloch equations or the density matrix equations for the evolution of the atomic variables in the rotating-wave approximation to have an invariant form for any energy-level ordering.

The evolution of the atomic system is described by the Liouville equation for the density matrix $\hat{\rho}(t)$ of the system:

$$i\hbar \frac{\partial \hat{\rho}(t)}{\partial t} = [\hat{H}(t), \hat{\rho}(t)], \tag{2}$$

where for problems in which a rotating-wave approximation is used and in which the dipole transition moments link the levels only stepwise, 1–2, 2–3, ..., $(N - 1) - N$, the matrix elements of the Hamiltonian $\hat{H}(t)$ for a particular atom can be written as

$$H_{nn}(t) = \hbar\Delta_n(t), \tag{3a}$$

$$H_{n,n+1}(t) = H^*_{n+1,n}(t) = -(1/2)\hbar\Omega_n(t), \tag{3b}$$

and

$$H_{nn'}(t) = 0 \text{ otherwise.} \tag{3c}$$

Here $\Delta_n(z, t)$, which generally depends on position and time, is the cumulative detuning of $n - 1$ successive laser frequencies from the corresponding sum of $n - 1$ Doppler-shifted transition frequencies, and $\Omega_n(z, t)$ is the appropriate Rabi frequency:

0740-3224/89/030335-09$02.00

$$\Omega_n = 2\hbar^{-1}(\mathbf{d}_{n,n+1} \cdot \mathbf{e}_n)\mathcal{E}_n(z, t), \tag{4}$$

where $\mathbf{d}_{n,n+1} = \langle n|\mathbf{d}|n + 1\rangle$ is the dipole moment between levels n and $n + 1$.

Equation (2), combined with the $N - 1$ reduced Maxwell equations given by

$$\left[\frac{\partial}{\partial z} + \frac{\partial}{\partial(ct)}\right]\Omega_n(z, t) = -i\,\frac{4\pi D}{\hbar c}\,C_n\nu_n d_n^{\,2}\langle\rho_{n,n+1}\rangle, \tag{5}$$

where $d_n = |\mathbf{d}_{n,n+1} \cdot \mathbf{e}_n|$ is the absolute value of the appropriate component of the dipole matrix element, D is the atomic density, and $\langle . \rangle$ denotes averaging over the Maxwellian velocity distribution of atoms, constitutes a semiclassical description of $N - 1$ electromagnetic pulses propagating in an atomic or molecular medium, with pulse lengths short compared with atomic or molecular relaxation times.

We shall now assume that the detunings $\Delta_n(t)$ and Rabi frequencies $\Omega_n(t)$ in Eqs. (3) are chosen to satisfy the following relations:

$$\Delta_n(t) = -[n - (1/2)(N + 1)]\Delta_0(t), \qquad n = 1, 2, \ldots, N \tag{6a}$$

and

$$\Omega_n(t) = \sqrt{n(N - n)}\,\Omega_0(t), \qquad n = 1, 2, \ldots, N - 1, \tag{6b}$$

where $\Delta_0(t)$ and $\Omega_0(t)$ can be arbitrary functions of time. then the matrix elements of the Hamiltonian are given by

$$H_{mm}(t) = -\hbar\Delta_0(t), \tag{10a}$$

$$H_{m,m+1}(t) = H^*_{m+1,m}(t) = -(1/2)[(j - m)(j + m + 1)]^{1/2}\hbar\Omega_0(t), \tag{10b}$$

$$H_{mm'}(t) = 0 \text{ otherwise}. \tag{10c}$$

As was pointed out in Refs. 7 and 10, the special symmetry feature of the SU(2) model in the Liouville Eq. (2) is best revealed if we use the irreducible tensorial sets of Racah[12–14] as the basis operators for the representation of $\hat{\rho}(t)$. In terms of the projection operator $|m\rangle\langle m'|$, the N^2 operators of Racah, $\hat{T}_q^{(k)}$, where for each $k = 0, 1, 2, \ldots, 2j$, $q = -k, -k + 1, \ldots, k$, are defined by

$$\hat{T}_q^{(k)} = \sum_{m,m'} (-1)^{j-m'}(2k + 1)^{1/2}\begin{pmatrix} j & k & j \\ -m' & q & m\end{pmatrix}|m'\rangle\langle m|, \tag{11}$$

where $m, m' = -j, -j + 1, \ldots, j$ and where

$$\begin{pmatrix} j_1 j_2 j_3 \\ m_1 m_2 m_3\end{pmatrix}$$

is the Wigner 3-j symbol.[11]

An expression for the purpose of computing the 3-j symbols here is the following:

$$\begin{pmatrix} j_1 j_2 j_3 \\ m_1 m_2 m_3\end{pmatrix} = \delta(m_1 + m_2 + m_3, 0)(-1)^{j_1 - j_2 - m_3}\left[\frac{(j_1 + j_2 - j_3)!(j_1 - j_2 + j_3)!(-j_1 + j_2 + j_3)!}{(j_1 + j_2 + j_3 + 1)!}\right]^{1/2}$$
$$\times [(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!(j_3 + m_3)!(j_3 - m_3)!]^{1/2}$$
$$\times \sum_s (-1)^s \frac{1}{s!(j_1 + j_2 - j_3 - s)!(j_1 - m_1 - s)!(j_2 + m_2 - s)!(j_3 - j_2 + m_1 + s)!(j_3 - j_1 - m_2 + s)!}.$$

For the special case when $\Delta_0(t)$ and $\Omega_0(t)$ are independent of the time or have the same time dependence, Eqs. (6) are known as the Cook–Shore condition.[8] More generally, an atomic or molecular system whose time-dependent Hamiltonian satisfies Eqs. (3) and (6) is said to possess the SU(2) dynamic symmetry,[9,10] and the model is called an N-level SU(2) model. Mathematically, the Hamiltonian of an N-level SU(2) model lies entirely in the subspace spanned by the generators of the $0^+(3)$ subgroup of the $N^2 - 1$ generators of the SU(2) algebra, i.e., it can be written as

$$\hat{H}(t) = c_1(t)\hat{J}_x + c_2(t)\hat{J}_y + c_3(t)\hat{J}_z + \mathrm{d}(t), \tag{7}$$

where $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ are the angular momentum operators of spin $j = (1/2)(N - 1)$. Since the presence of $\mathrm{d}(t)$ in Eq. (7) only changes the result for the probability amplitudes by the same phase factor, we shall ignore its presence. We use the representation in which $\hat{J}_z$ is diagonal; Eq. (7) gives Eqs. (3) and (6) if we identify

$$c_1(t) + ic_2(t) = \hbar\Omega_0(t), \qquad c_3(t) = -\hbar\Delta_0(t). \tag{8}$$

If we use the labels $m = -j, -j + 1, \ldots, j$ for the angular momentum j, which are related to the labels $n = 1, 2, \ldots, N$ for an N-level system by

$$m = n - j - 1, \qquad N = 2j + 1, \tag{9}$$

Note the relationships

$$(\hat{T}_q^{(k)})^\dagger = (-1)^q(\hat{T}_{-q}^{(k)}) \tag{12a}$$

and

$$\mathrm{tr}[(\hat{T}_q^{(k)})^\dagger\hat{T}_{q'}^{(k')}] = \delta(kk')\delta(qq'). \tag{12b}$$

In terms of this basis set, the density matrix $\hat{\rho}(t)$ has N^2 components $T_q^{(k)}(t)$, as expressed by

$$\hat{\rho}(t) = \sum_{k=0}^{2j}\sum_{q=-k}^{k} T_q^{(k)}(t)\hat{T}_q^{(k)}, \tag{13a}$$

where

$$T_q^{(k)}(t) = \mathrm{tr}[\hat{\rho}(t)(\hat{T}_q^{(k)})^\dagger]. \tag{13b}$$

Expressing the Liouville Eq. (2) in terms of the $\hat{T}_q^{(k)}(t)$, we find that

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}T_q^{(k)}(t) = \sum_{k',q'} A_{kq,k'q'}(t)T_{q'}^{(k')}(t), \tag{14a}$$

where

$$A_{kq,k'q'}(t) = -\mathrm{tr}\{\hat{H}(t)[(\hat{T}_q^{(k)})^\dagger, \hat{T}_{q'}^{(k')}]\}. \tag{14b}$$

At this point, the special feature of the SU(2) model, as expressed by its Hamiltonian given by Eqs. (10), will be noticed, for it can be shown[7,10] from Eq. (14b) that $A_{kq,k'q'}(t) = 0$ unless $k = k'$. That is to say, the N^2-dimensional space in which the atomic variables evolve decomposes, in the basis set of Racah, into N independent subspaces of dimensions $2k + 1$, with $k = 0, 1, 2, \ldots, 2j$, i.e., of dimensions $1, 3, 5, \ldots, 4j + 1$. In each subspace characterized by the value of k, the equation of motion for $T_q^{(k)}(t)$ is given by[15]

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}^{(k)}(t) = \hat{A}(t)\mathbf{T}^{(k)}(t), \tag{15}$$

where the matrix elements of the matrix $\hat{A}(t)$ are given by

$$A_{qq}(t) = -q\hbar\Delta_0(t), \tag{16a}$$

$$A_{q,q+1}(t) = A^*_{q+1,q}(t) = -(1/2)[(k-q)(k+q+1)]^{1/2}\hbar\Omega_0(t), \tag{16b}$$

$$A_{qq'}(t) = 0 \text{ otherwise}, \tag{16c}$$

where $k = 0, 1, 2, \ldots, 2j$ and $q = -k, -k + 1, \ldots, k$.

Notice that Eqs. (16) exactly parallel Eqs. (10) except that k takes up only integer values here while j in Eqs. (10) takes up half-integer as well as integer values.

Following the solution given by me in Ref. 9, the solution of Eq. (15) can be readily written down. Let $a(t)$ and $-b^*(t)$ denote the solutions for $C_1(t)$ and $C_2(t)$, respectively, satisfying the following equations of motion:

$$i\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{bmatrix} (1/2)\Delta_0(t) & -(1/2)\Omega_0(t) \\ -(1/2)\Omega_0^*(t) & -(1/2)\Delta_0(t) \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \tag{17}$$

with the initial conditions $|C_1(0)| = 1$, $C_2(0) = 0$, so that the solution of Eqs. (17) for the general initial values of $C_1(0)$ and $C_2(0)$ can be written as

$$\begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} = \begin{bmatrix} a(t) & b(t) \\ -b^*(t) & a^*(t) \end{bmatrix}\begin{bmatrix} C_1(0) \\ C_2(0) \end{bmatrix}. \tag{18}$$

Then for each value of k in Eq. (15) the solution for $T_q^{(k)}(t)$ is given by

$$T_q^{(k)}(t) = \sum_{q'=-k}^{k} D_{qq'}^{(k)}(a, b)T_{q'}^{(k)}(0), \qquad q = -k, -k+1, \ldots, k, \tag{19}$$

where $D_{qq'}^{(k)}(a, b)$ are the matrix elements of the $(2k + 1)$-dimensional representation of the SU(2) group.[16] The following expression:

$$D_{mm'}^{(j)}(a, b) = \sum_{\mu} \frac{[(j-m)!(j+m)!(j-m')!(j+m')!]^{1/2}}{p!q!r!s!} \times a^p a^{*q} b^r (-b^*)^s, \tag{20}$$

where (i) $p = j - m - \mu$, $q = j + m' - \mu$, $r = \mu$, $s = m - m' + \mu$, (ii) $p = m - m' + \mu$, $q = \mu$, $r = j + m' - \mu$, $s = j + m - \mu$, or (iii) $p = \mu$, $q = m + m' + \mu$, $r = j - m - \mu$, $s = j - m' - \mu$, and where $\mu = 0, 1, 2, \ldots$ can be used, all will give the same result. $T_q^{(k)}(t)$ having been obtained from Eqs. (19) and (17), the density matrix $\hat{\rho}(t)$ is determined from Eqs. (13a) and (11).

Having obtained the complete solution of Eq. (2) for the time evolution of a specific atom in the atomic medium for any initial condition, our next task will be to use it to solve the reduced Maxwell equations (5) for a set of pulses propagating through the atomic medium. An analytic solution of the problem under a general condition can be seen to be quite complicated and may not even be useful. On the other hand, our solution in the form of Eq. (13a) can be used, as will be shown, to deduce the special conditions for the simulton propagation.

We first make use of the facts that the atomic variables $T_q^{(k)}(t)$ in different subspaces characterized by different values of k evolve independently and that the solution for $\hat{\rho}(t)$, Eq. (13a), is a superposition of these independent solutions. Thus if we choose the initial values of $\rho_{nn'}(0)$ in such a way that $T_q^{(k)}(0) = 0$ except for $k = 0$ and $k = k_1 \neq 0$, then the atomic variables in the other subspaces will remain zero at all times, i.e., $T_q^{(k)}(t) = 0$ except for $k = 0$ and $k = k_1$, for which

$$T_0^{(0)}(t) = T_0^{(0)}(0) = (2j + 1)^{-1/2} \tag{21}$$

and

$$T_q^{(k_1)}(t) = \sum_{q'=-k_1}^{k_1} D_{qq'}^{(k_1)}(a, b)T_{q'}^{(k_1)}(0), \tag{22}$$

and hence

$$\hat{\rho}(t) = (2j + 1)^{-1}\hat{I} + \sum_{q=-k_1}^{k_1} T_q^{(k_1)}(t)\hat{T}_q^{(k_1)}, \tag{23}$$

where $\hat{I}$ denotes a unit matrix. Let us assume that the initial values of the off-diagonal elements are zero, i.e., that $\rho_{nn'}(0) = 0$ for $n \neq n'$. Then the requirement that $T_q^{(k)}(0) = 0$ except for $k = 0$ and $k = k_1 \neq 0$ means that the initial values of the diagonal elements $\rho_{nn}(0)$ or the initial-level population should be

$$\rho_{mm}(0) = (2j + 1)^{-1} + (-1)^{j-m}(2k_1 + 1)^{1/2} \times \begin{pmatrix} j & k_1 & j \\ -m & 0 & m \end{pmatrix} T_0^{(k_1)}(0), \tag{24}$$

where $T_0^{(k_1)}(0)$ is an arbitrary real constant that may be positive or negative subject only to the condition that it should make

$$\rho_{mm}(0) \geqq 0 \qquad \text{for all } m = -j, -j + 1, \ldots, j. \tag{25}$$

This can be shown as follows. From Eq. (13b) we find that

$$T_q^{(k)}(t) = \sum_m (-1)^{j-m+q}(2k + 1)^{1/2} \times \begin{pmatrix} j & k & j \\ -m-q & q & m \end{pmatrix} \rho_{m+q,m}(t). \tag{26}$$

If $\rho_{mm'}(0) = 0$ for $m \neq m'$, and $\rho_{mm}(0)$ is given by Eq. (24), then substituting Eq. (24) into Eq. (26) gives, for $k \neq 0$,

$$T_q^{(k)}(0) = 0 \qquad \text{for } q \neq 0 \tag{27a}$$

and

$$T_0^{(k)}(0) = T_0^{(k_1)}(0)\delta(k, k_1), \tag{27b}$$

where we have made use of the orthogonality property of the 3-j symbols[11]:

$$\sum_m \begin{pmatrix} j & k & j \\ -m-q & q & m \end{pmatrix}\begin{pmatrix} j & k_1 & j \\ -m-q & q & m \end{pmatrix} = (2k_1+1)^{-1}\delta(k, k_1) \quad (28)$$

and the relation

$$\begin{pmatrix} j & 0 & j \\ -m & 0 & m \end{pmatrix} = (-1)^{j-m}(2j+1)^{-1/2}. \quad (29)$$

The $N-1$ sets of initial-level population given by setting $k_1 = 1, 2, \ldots, N-1$ successively in Eq. (24) are independent and orthogonal in the sense that any given arbitrary distribution of initial-level population $\hat{\rho}(0)$ can be expressed as a superposition of these sets:

$$\rho_{mm}(0) = (2j+1)^{-1} + \sum_{k_1=1}^{2j} (-1)^{j-m}(2k_1+1)^{1/2} \times \begin{pmatrix} j & k_1 & j \\ -m & 0 & m \end{pmatrix} T_0^{(k_1)}(0)|m\rangle\langle m|, \quad (30)$$

where

$$T_0^{(k_1)}(0) = \sum_m \rho_{mm}(0)(-1)^{j-m}(2k_1+1)^{1/2}\begin{pmatrix} j & k_1 & j \\ -m & 0 & m \end{pmatrix}. \quad (31)$$

The special choice of the initial-level population as given by Eq. (24) in which we set $k_1 = 1, 2, \ldots, N-1$ successively greatly simplifies our expressions for the atomic variables for each case. For each value of k_1, we obtain $T_q^{(k_1)}(t)$ from Eq. (22) and then obtain $\rho_{m+q,m}(t)$ and $\rho_{m,m+q}(t)$ from

$$\rho_{m+q,m}(t) = (2j+1)\delta_{q,0} + (-1)^{j-m+q}(2k_1+1)^{1/2} \times \begin{pmatrix} j & k_1 & j \\ -m-q & q & m \end{pmatrix} T_q^{(k_1)}(t) \quad (32a)$$

and

$$\rho_{m,m+q}(t) = (2j+1)\delta_{q,0} + (-1)^{j-m}(2k_1+1)^{1/2} \times \begin{pmatrix} j & k_1 & j \\ -m-q & q & m \end{pmatrix} T_{-q}^{(k_1)}(t). \quad (32b)$$

The atomic variables that we require for the reduced Maxwell Eqs. (5) are $\rho_{m,m+1}(t)$, given by

$$\rho_{m,m+1}(t) = (-i)(-1)^{j-m}(2k_1+1)^{1/2} \times \begin{pmatrix} j & k_1 & j \\ -m-1 & 1 & m \end{pmatrix} T_{-1}^{(k_1)}(t). \quad (33)$$

Since the $\Omega_m(z, t)$ in Eq. (5) are already chosen to be given by

$$\Omega_m(z, t) = [(j-m)(j+m+1)]^{1/2}\Omega_0(z, t), \quad (34)$$

the set of Eqs. (5) with $\rho_{m,m+1}(t)$ given by Eqs. (33) substituted into them would give a consistent result if the following relations are satisfied:

$$\frac{C_m \nu_m \mathrm{d}_{m,m+1}^2}{[(j-m)(j+m+1)]^{1/2}}(-1)^{j-m}\begin{pmatrix} j & k_1 & j \\ -m-1 & 1 & m \end{pmatrix} = \text{const.} \quad (35)$$

for $m = -j, -j+1, \ldots, j-1$. When Eqs. (35) are satisfied, the $N-1$ reduced Eqs. (5) reduce to a single equation for $\Omega_0(z, t)$, which is the common factor appearing in the $N-1$ Rabi frequencies [Eq. (6b)], given by

$$\left[\frac{\partial}{\partial z} + \frac{\partial}{\partial(ct)}\right]\Omega_0(z, t) = (-i)\frac{4\pi D}{\hbar c}(2k_1+1)^{1/2}D_{-1,0}^{(k_1)}(a, b) \times T_0^{(k_1)}(z, 0)(-1)^{j-m}\frac{\begin{pmatrix} j & k_1 & j \\ -m-1 & 1 & m \end{pmatrix}}{[(j-m)(j+m+1)]^{1/2}} C_m \nu_m d_m^2, \quad (36)$$

where $a(z, t)$ and $-b^*(z, t)$ are the solutions for C_1 and C_2 of Eqs. (17) with $\Delta_0(z, t)$ and $\Omega_0(z, t)$ replacing $\Delta_0(t)$ and $\Omega_0(t)$, respectively.

The two conditions [Eqs. (24) and (35)] can be combined to give the following simpler relation:

$$C_m \nu_m d_m^2[\rho_{mm}(0) - \rho_{m+1,m+1}(0)] = \text{const. (positive)} \quad (37)$$

by using the following relation among the 3-j symbols[17]:

$$\begin{pmatrix} j & k & j \\ -m-1 & 0 & m+1 \end{pmatrix} + \begin{pmatrix} j & k & j \\ -m & 0 & m \end{pmatrix} = -\frac{[k(k+1)]^{1/2}}{[(j-m)(j+m+1)]^{1/2}}\begin{pmatrix} j & k & j \\ -m-1 & 1 & m \end{pmatrix}. \quad (38)$$

It is interesting that condition (37), in addition to being mathematically compact, also has a direct physical interpretation, which suggested its significance in advance[18]: In laser theory the gain on a lasing transition is directly proportional to the product of factors given in relation (37). Thus relation (37) can be interpreted as an equal-gain condition for all transitions in order that simulton propagation be supported by the atomic system.

Equations (36) can now be written as

$$\left[\frac{\partial}{\partial z} + \frac{\partial}{\partial(ct)}\right]\Omega_0(z, t) = -\frac{2\pi D}{\hbar c} C_m \nu_m d_m^2 \times [\rho_{mm}(0) - \rho_{m+1,m+1}(0)] f^{(k_1)}(a, b), \quad (39)$$

where

$$f^{(k_1)}(a, b) = \frac{-2iD_{-1,0}^{(k_1)}(a, b)}{[k_1(k_1+1)]^{1/2}}. \quad (40)$$

Let

$$\zeta = t - z/V, \quad (41)$$

where V denotes the velocity of the pulse, so that

$$\frac{\partial}{\partial t} \to \frac{\mathrm{d}}{\mathrm{d}\zeta}, \qquad \frac{\partial}{\partial z} \to -\frac{1}{V}\frac{\mathrm{d}}{\mathrm{d}\zeta}, \quad (42)$$

and let

$$\Theta(z, t) = \int_{-\infty}^{t} \Omega_0(z, t')\mathrm{d}t'. \quad (43)$$

Then Eq. (39) can be written as

$$\ddot{\Theta} = \frac{1}{\tau^2} f^{(k_1)}(a, b), \quad (44)$$

where the dot denotes the derivative with respect to ζ, where

$$\frac{1}{\tau^2} = \frac{2\pi D}{\hbar c\left(\frac{1}{V} - \frac{1}{c}\right)} C_m d_m{}^2 \nu_m [\rho_{mm}(0) - \rho_{m+1,m+1}(0)], \quad (45)$$

and where $f^{(k_1)}(a, b)$ is given by Eq. (40). The specific examples that we shall give below will show that a solitary-pulse solution for Eq. (44) requires that $1/\tau^2$ be positive.

Equations (24) and (35) constitute the two conditions that must be satisfied so that the equation for the propagation of the common pulse amplitude specified by $\Omega_0(z, t)$ is determined by Eq. (44), which, as we shall see, permits special solitary-pulse solutions.

More specifically, the necessary conditions for the initial-level population as well as for the frequencies (ν_m) of the lasers, the dipole moment matrix elements (d_m), and the level configuration (C_m), specified by Eqs. (24) and (37), are, for

$$k_1 = 1, \quad \rho_{mm}(0) = m\delta + (2j+1)^{-1}, \quad C_m \nu_m d_m{}^2 = \text{const.}, \quad (46a)$$

$$k_1 = 2, \quad \rho_{mm}(0) = [3m^2 - j(j+1)]\delta + (2j+1)^{-1},$$
$$C_m \nu_m d_m{}^2 (2m+1) = \text{const.}, \quad (46b)$$

$$k_1 = 3, \quad \rho_{mm}(0) = [5m^3 - (3j^2 + 3j - 1)m]\delta + (2j+1)^{-1},$$
$$C_m \nu_m d_m{}^2 [2 + 5m(m+1) - j(j+1)] = \text{const.}, \quad (46c)$$

$$k_1 = 4, \quad \rho_{mm}(0) = \{70m^4 - [60j(j+1) - 50]m^2 + 6(j-1)j(j+1)(j+2)\}\delta + (2j+1)^{-1},$$
$$C_m \nu_m d_m{}^2 \{14m^3 + 21m^2 + [19 - 6j(j+1)]m + [6 - 3j(j+1)]\} = \text{const.}, \quad (46d)$$

where δ can be positive or negative subject only to the requirement that $\rho_{mm}(0) \geq 0$ for all $m = -j, -j+1, \ldots, j$ and where $j = (1/2)(N-1)$.

For example, for a five-level ($N = 5$ or $j = 2$) system whose Hamiltonian satisfies Eqs. (5) and (6) to support the simulton propagation, the conditions expressed by Eqs. (24) and (37), as well as the necessary level configurations or ordering required by $1/\tau^2$ being positive in Eq. (45), for the four mutually exclusive cases characterized by $k_1 = 1, 2, 3, 4$ are given explicitly in Table 1, where $\rho_{\bar{j}\bar{j}}(0)$ denotes $\rho_{jj}(0) - (1/5)$. The corresponding level configurations are sketched in Figs. 1(a), 1(b), 1(c), and 1(d), respectively.

The quantity $f^{(k_1)}(a, b)$ given by Eq. (40) can be expressed in terms of the Jacobi polynomial[19] $P_n{}^{(\alpha,\beta)}(x)$ as follows:

$$f^{(k_1)}(a, b) = \frac{2i}{k_1 + 1} ab|b|^{-2} P_{k_1}{}^{(-1,1)}(x), \quad (47)$$

where

$$x = |a|^2 - |b|^2. \quad (48)$$

By using the expansion

$$P_n{}^{(\alpha,\beta)}(x) = \sum_{m=0}^{n} (-1)^{n-m} \binom{n+\alpha}{m} \binom{n+\beta}{n-m} \times \left(\frac{1-x}{2}\right)^{n-m} \left(\frac{1+x}{2}\right)^m \quad (49)$$

and the relation $|a|^2 P_n{}^{(-1,1)}(x) = |b|^2 P_n{}^{(1,-1)}(x)$, where

$$|a|^2 = (1/2)(1+x), \qquad |b|^2 = (1/2)(1-x), \quad (50)$$

Eq. (47) can be written as

$$f^{(k_1)}(a, b) = \frac{i}{k_1 + 1} ab\{[(1/2)(1-x)]^{-1} P_{k_1}{}^{(-1,1)}(x) - [(1/2)(1+x)]^{-1} P_{k_1}{}^{(1,-1)}(x)\}. \quad (51)$$

We can now use the recurrence relation[20]

$$(1+x)P_n{}^{(-1,1)}(x) - (1-x)P_n{}^{(1,-1)}(x) = -\frac{2}{n}(1-x^2)P_n{}'(x),$$

where $P_n(x)$ is the Legendre polynomial of order n and where the prime denotes the derivative with respect to x, to simplify Eq. (51) further to

$$f^{(k_1)}(a, b) = -\frac{4i}{k_1(k_1+1)} abP_{k_1}{}'(x). \quad (52)$$

Specifically, we find, for

$$k_1 = 1, \quad f^{(1)}(a, b) = -2iab,$$

$$k_1 = 2, \quad f^{(2)}(a, b) = -2iab(|a|^2 - |b|^2),$$

$$k_1 = 3, \quad f^{(3)}(a, b) = -2iab(|a|^4 - 3|a|^2|b|^2 + |b|^4),$$

$$k_1 = 4, \quad f^{(4)}(a, b) = -2iab(|a|^2 - |b|^2)(|a|^4 - 5|a|^2|b|^2 + |b|^4).$$

Substituting Eq. (52) into Eq. (44) gives

$$\ddot{\theta} = -\frac{4i}{\tau^2 k_1(k_1+1)} abP_{k_1}{}'(x). \quad (53)$$

We now restrict our consideration to the case for which $\Omega_0(z, t)$ is real and $\Delta_0(z, t) = 0$. $a(z, t)$ and $b(z, t)$ can be expressed in this case in terms of an angle $\theta(z, t)$ as

$$a(z, t) = \cos(1/2)\theta, \quad (54a)$$

Table 1. Necessary Conditions for a Five-Level System To Support the Simulton Propagation[a]

	Initial-Level Population					
	$\rho_{\bar{1}\bar{1}}(0)$	$\rho_{\bar{2}\bar{2}}(0)$	$\rho_{\bar{3}\bar{3}}(0)$	$\rho_{\bar{4}\bar{4}}(0)$	$\rho_{\bar{5}\bar{5}}(0)$	Condition for ν_j and d_j
$k_1 = 1$	6δ	3δ	0	-3δ	-6δ	$\nu_1 d_1{}^2 = \nu_2 d_2{}^2 = \nu_3 d_3{}^2 = \nu_4 d_4{}^2$
$k_1 = 2$	6δ	-3δ	-6δ	-3δ	6δ	$3\nu_1 d_1{}^2 = \nu_2 d_2{}^2 = \nu_3 d_3{}^2 = 3\nu_4 d_4{}^2$
$k_1 = 3$	3δ	-6δ	0	6δ	-3δ	$3\nu_1 d_1{}^2 = 2\nu_2 d_2{}^2 = 2\nu_3 d_3{}^2 = 3\nu_4 d_4{}^2$
$k_1 = 4$	δ	-4δ	6δ	-4δ	δ	$\nu_1 d_1{}^2 = 2\nu_2 d_2{}^2 = 2\nu_3 d_3{}^2 = \nu_4 d_4{}^2$

[a] $\rho_{\bar{j}\bar{j}}(0)$ denotes $\rho_{jj}(0) - (1/5)$, and $0 < \delta < 1/30$.

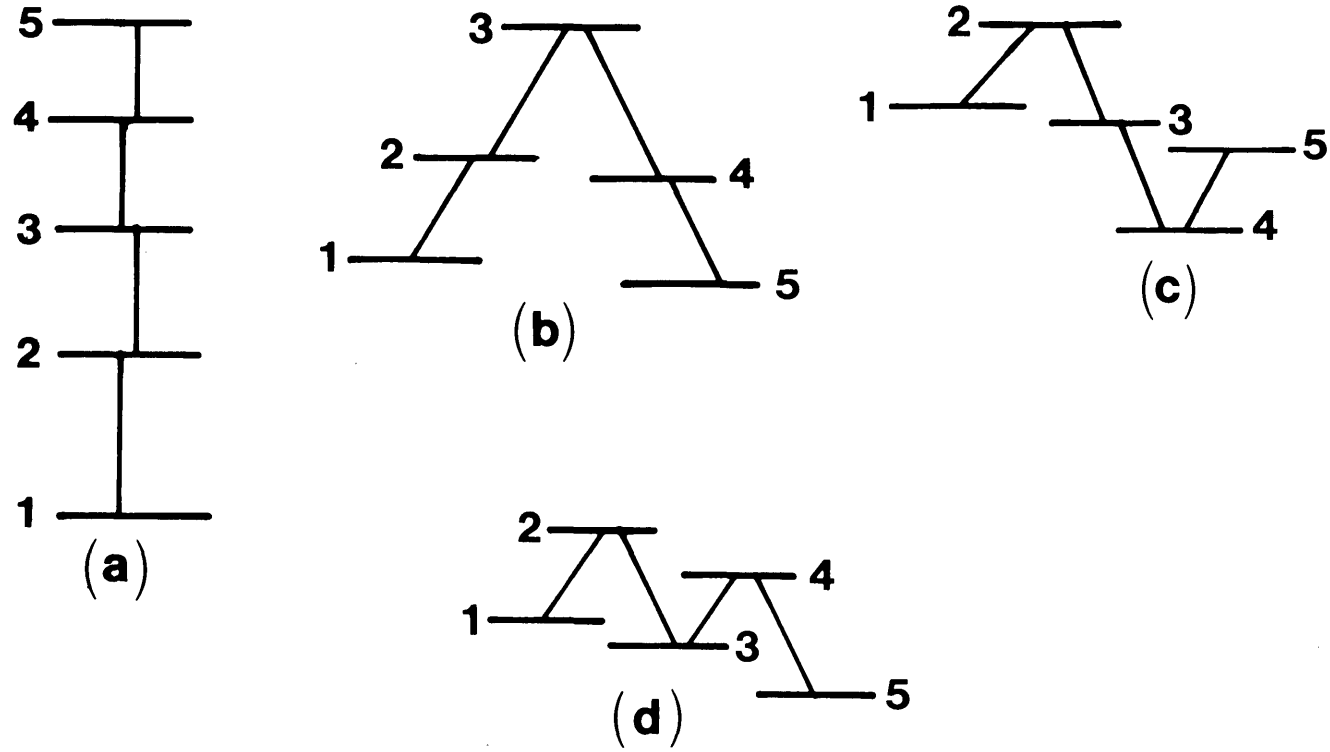

Fig. 1. (a), (b), (c), (d) Energy-level configurations for the five-level system for the cases k_1 equal to 1, 2, 3, and 4, respectively, given in Table 1.

$$b(z, t) = i \sin(1/2)\theta, \tag{54b}$$

so that

$$-2iab = \sin\theta, \tag{55}$$

$$|a|^2 - |b|^2 = \cos\theta, \tag{56}$$

$$f^{(k_1)}(\theta) = \frac{-2}{k_1(k_1+1)} \frac{\mathrm{d}}{\mathrm{d}\theta} P_{k_1}(\cos\theta), \tag{57}$$

and Eq. (53) becomes

$$\ddot{\Theta} = \frac{-2}{\tau^2 k_1(k_1+1)} \frac{\mathrm{d}}{\mathrm{d}\theta} P_{k_1}(\cos\theta). \tag{58}$$

The two angles, θ and Θ, given by Eqs. (54) and (43), respectively, are identical, as can be seen from the solution of the two-level system, Eqs. (17), when $\Omega_0(z, t)$ is real and $\Delta_0(z, t) = 0$. Thus Eq. (58) becomes

$$\ddot{\theta} = \frac{1}{\tau^2} f^{(k_1)}(\theta), \tag{59}$$

where from Eq. (57) we find that[21]

$$f^{(k_1)}(\theta) = \begin{cases} \displaystyle\sum_{r=0}^{n} c_{2n-2r+1} \sin(2n-2r+1)\theta & \text{for } k_1 = 2n+1 \quad (60a) \\ \displaystyle\sum_{r=0}^{n-1} c_{2n-2r} \sin(2n-2r)\theta & \text{for } k_1 = 2n \quad (60b) \end{cases},$$

where

$$c_{2n-2r+1} = \frac{1}{k_1(k_1+1)} \frac{1}{2^{4n}} \frac{(4n+2-2r)!(2r)!}{(r!)^2[(2n+1-r!)]^2} (2n-2r+1) \tag{61a}$$

and

$$c_{2n-2r} = \frac{1}{k_1(k_1+1)} \frac{1}{2^{4n-2}} \frac{(4n-2r)!(2r)!}{(r!)^2[(2n-r)!]^2} (2n-2r). \tag{61b}$$

The following expansion[22] for the Legendre polynomial can be used to arrive at Eqs. (60) and (61) from Eq. (58):

$$P_n(\cos\theta) = \sum_{m=0}^{n} a_m \cos(n-2m)\theta \tag{62}$$

and

$$a_m = \frac{1}{4^n} \binom{2m}{m} \binom{2n-2m}{n-m}.$$

Equations (60) and (61) show that $f^{(k)}(\theta)$ is a sum of $\sin k\theta$, $\sin(k-2)\theta, \ldots,$ with *positive* coefficients. For example, for $k = 1$–4, $f^{(k)}(\theta)$ are given by

$$f^{(1)}(\theta) = \sin\theta, \tag{63a}$$

$$f^{(2)}(\theta) = (1/2)\sin 2\theta, \tag{63b}$$

$$f^{(3)}(\theta) = (1/16)(\sin\theta + 5\sin 3\theta), \tag{63c}$$

$$f^{(4)}(\theta) = (1/32)(2\sin 2\theta + 7\sin 4\theta). \tag{63d}$$

Equation (59) can be readily integrated to give

$$\dot{\theta}^2 = \frac{4}{\tau^2 k_1(k_1+1)}[1 - P_{k_1}(\cos\theta)], \tag{64}$$

where we have assumed that $\theta = \dot{\theta} = 0$ at the initial time.

When the electric-field envelope area is defined as in Eq. (43),

$$A(z, t) = \int_{-\infty}^{t} \Omega_0(z, t')\mathrm{d}t', \tag{65}$$

the corresponding area theorem of McCall and Hahn[23] is

$$\frac{\partial}{\partial z} A(z, t) = -(1/2)\alpha f^{(k_1)}(A), \tag{66}$$

where α is the absorption coefficient and where $f^{(k_1)}(A)$ is given as in Eqs. (57) and (60). Thus, if $A = n\pi$ for any positive integer n, the pulse envelope area suffers no attenuation in propagation since $\partial A/\partial z = 0$. The areas that are even multiples of π are known to be more stable than those that are odd multiples.[3] These pulses travel anomalously slowly but behave as if the medium were transparent in the sense that they suffer no attenuation during propagation.

Equation (64) shows that the simulton propagation problem can be reduced to quadratures, given the conditions for any k_1. If we set $x = \cos\theta$ in Eq. (64), the integrated solution is given by

$$\int^{x} \frac{\mathrm{d}x}{\{(1-x^2)[1-P_{k_1}(x)]\}^{1/2}} = -\frac{2}{\tau[k_1(k_1+1)]^{1/2}}(\zeta - \zeta_0).$$

The integral on the left-hand side can be expressed in terms of elementary functions or elliptic integrals for the cases $k_1 =$ 1–4 and $k_1 = 6$. The explicit analytic expressions for θ in terms of $\zeta - \zeta_0$ are simple for the first two cases ($k_1 = 1, 2$). For $k_1 = 1$, Eq. (59) becomes

$$\ddot{\theta} = \frac{1}{\tau^2}\sin\theta, \tag{67}$$

and the solitary-pulse solution is

$$\dot{\theta} = \Omega_0(\zeta) = \frac{2}{\tau}\operatorname{sech}\frac{\zeta - \zeta_0}{\tau} \tag{68a}$$

or

$$\theta = 4\tan^{-1}\left(\exp\frac{\zeta - \zeta_0}{\tau}\right). \tag{68b}$$

We have assumed that Eqs. (46a) are satisfied. In addition, the requirement that $1/\tau^2$ be positive in Eq. (45) requires that $C_m[\rho_{mm}(0) - \rho_{m+1,m+1}(0)]$ be positive, and this can be satisfied if the levels are of the cascade configuration, i.e., $E_{n+1} > E_n$ for all n or $E_{n+1} < E_n$ for all n, with the level population arranged accordingly [Eqs. (46a)]. The simultons consisting of $N - 1$ solitary pulses possibly of different frequencies but of the same speed V are given, from Eqs. (6), by

$$\Omega_n(\zeta) = \sqrt{n(N-n)}\,\Omega_0(\zeta), \qquad n = 1, 2, \ldots, N-1. \tag{69}$$

This is the simulton solution given in Refs. 1 and 2. For $k_1 =$ 2, Eq. (59) becomes

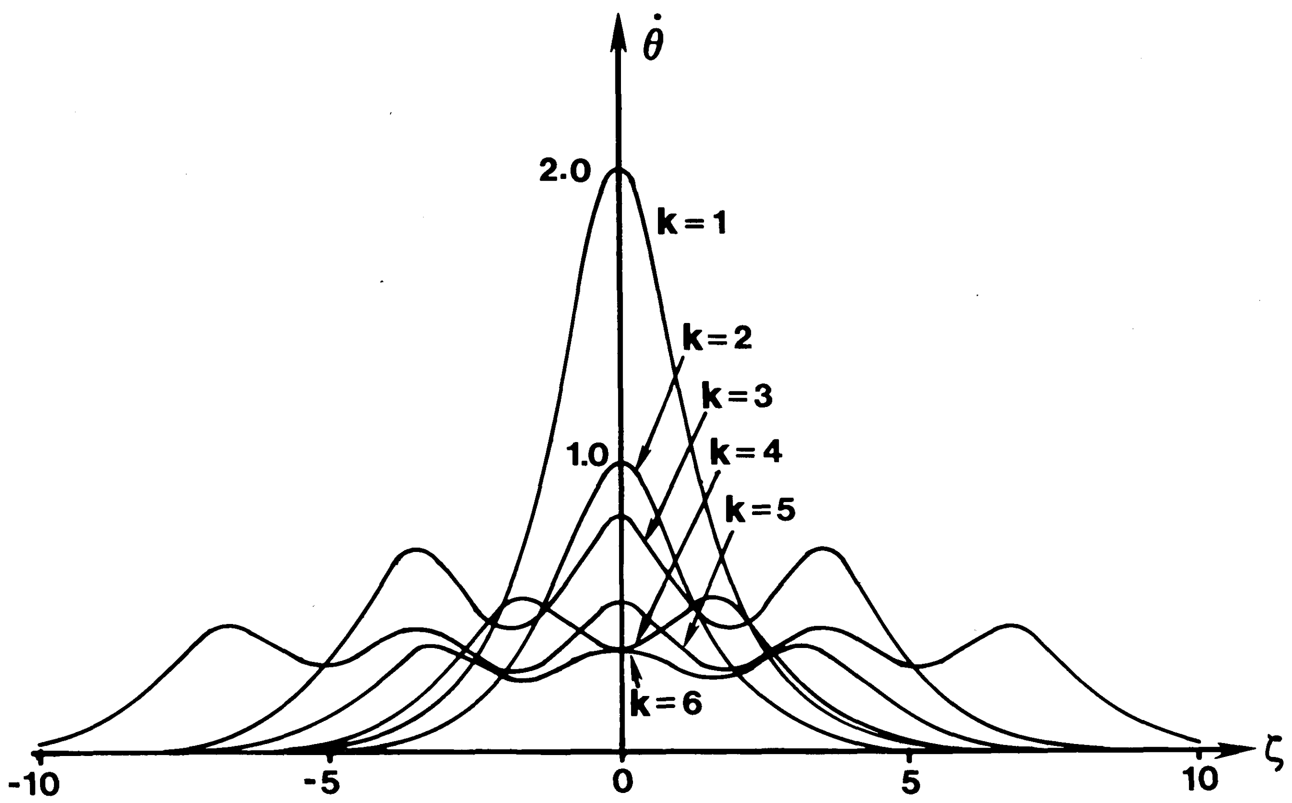

Fig. 2. Solitary-pulse solutions $\dot{\theta}$ of Eq. (59) or (64) as a function of ζ (setting $\zeta_0 = 0$ and $\tau = 1$) for the cases $k = k_1 = 1$–6. The area under each pulse is equal to 2π for k odd and to π for k even.

$$2\ddot{\theta} = \frac{1}{\tau^2}\sin 2\theta, \tag{70}$$

and the solitary-pulse solution is

$$\dot{\theta} = \Omega_0(\zeta) = \frac{1}{\tau}\operatorname{sech}\frac{\zeta - \zeta_0}{\tau} \tag{71a}$$

or

$$\theta = 2\tan^{-1}\left(\exp\frac{\zeta - \zeta_0}{\tau}\right). \tag{71b}$$

We have assumed in this case that Eqs. (46b) are satisfied. In addition, the requirement that $1/\tau^2$ be positive in Eq. (45) requires that the number of levels be odd, with the energies $E_{n+1} > E_n$ (or $E_{n+1} < E_n$) for $n = 1, 2, \ldots, (N-1)/2$, and $E_{n+1} < E_n$ (or $E_{n+1} > E_n$) for $n = (N+1)/2, (N+1)/2 + 1, \ldots, N$. This is the simulton solution given in Ref. 6.

For $k_1 > 2$, the solitary-pulse solution of Eq. (59) or (64) cannot be expressed simply. Instead, we shall examine the shapes of these solitary pulses numerically computed from Eq. (64) and point out their general features. They are presented in Fig. 2 for $k_1 = 3$–6 together with the familiar hyperbolic-secant pulse for $k_1 = 1, 2$. It will be noted that the area of the pulse given by

$$A = \lim_{t\to+\infty}\theta(t)$$

is

$$A = \begin{cases} 2\pi & \text{for } k_1 \text{ odd} \\ \pi & \text{for } k_1 \text{ even} \end{cases}, \tag{72}$$

which can be understood from Eq. (64) by noting that $P_n(-1) = (-1)^n$ and $P_n(1) = 1$, and hence $\dot{\theta}$, starting from zero, approaches its next zero as θ approaches 2π and π for k_1 odd and k_1 even, respectively. The solitary pulse has k_1 maxima and $k_1 - 1$ minima if k_1 is odd and has $k_1/2$ maxima and $k_1/2 - 1$ minima if k_1 is even. In terms of $\theta(t)$, which is given by

$$\theta(t) = \int_{-\infty}^{t}\dot{\theta}(t)\mathrm{d}t, \tag{73}$$

the maxima or minima of $\dot{\theta}(t)$ as a function of t occur at the values θ_m given by

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{k_1}(\cos\theta) = 0. \tag{74}$$

Since the pulse is symmetrical about $t = t_0$ (which may be taken to be zero), for k_1 odd, $t = t_0$ corresponds to $\theta = \pi$, and it always gives a maximum of $\dot{\theta}$; but for k_1 even, $t = t_0$ corresponds to $\theta = \pi/2$, and it gives a maximum of $\dot{\theta}$ if $k_1/2$ is an odd integer and a minimum of $\dot{\theta}$ if $k_1/2$ is an even integer. The value of $\dot{\theta}$ at the maximum or minimum corresponding to $\theta = \theta_m$ given by Eq. (74) is

$$\dot{\theta}|_{\theta=\theta_m} = \left\{\frac{4}{\tau^2 k_1(k_1+1)}[1 - P_{k_1}(\cos\theta_m)]\right\}^{1/2}. \tag{75}$$

It is useful to note that

$$P_n(0) = \begin{cases} 0 & \text{for } n \text{ odd} \\ \dfrac{(-1)^{(1/2)n}n!}{2^n[(n/2)!]^2} & \text{for } n \text{ even} \end{cases}. \tag{76}$$

At $t \to \pm\infty$, $\dot{\theta}(t)$ approaches zero exponentially. Since the pulse area $A = n\pi$, it is seen from Eqs. (60) and (66) that the pulse envelope area will suffer no attenuation in propagation.

To summarize our results, we began with an atomic medium consisting of identical atoms each of which had generally N transition levels that were chainwise dipole connected, and we considered sending $N - 1$ simultaneous equal-velocity laser pulses of possibly different wavelengths through the medium. We assumed that the time-dependent Hamiltonian of the laser–atom interacting system satisfied Eqs. (10). The system was said to possess the SU(2) dynamic symmetry. We assumed that Eqs. (24) and (37) were satisfied. That means that, given an atomic medium for which the $N - 1$ dipole moments d_m of the N transition levels of each atom were given, the level configuration C_m was assumed to be one of the $N - 1$ possibilities (see the example for $N = 5$ given in Fig. 1), and the laser frequencies ν_m and the initial level population $\rho_{mm}(0)$ were assumed to have been chosen appropriately so that Eqs. (24) and (37) were satisfied, the order k_1 of the possible simulton solution being determined by the level configuration of the atoms. When the above conditions are satisfied, simultons (of order k_1) consisting of $N - 1$ $\Omega_n(\zeta)$ given by Eq. (69), where $\Omega_0(\zeta)$ or $\dot{\theta}$ is given by the solution of Eq. (59) or (64), can propagate through the atomic medium without attenuation.

ACKNOWLEDGMENTS

This research is supported in part by the U.S. Department of Energy, Office of Basic Energy Sciences, Division of Chemical Sciences, under grant DE-FG02-84-ER 13243. I am particularly grateful to J. H. Eberly, who foresaw the relation given by Eq. (37) and its physical significance. Its use led to many simplifications of the subsequent expressions. I am also grateful to Galen Pickett for his computational help in producing Fig. 2 and to Daniel Koltun for telling me about Ref. 17. I thank the referees for their careful reading of the manuscript and for many helpful comments.

The author is also with the Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627.

Note Added in Proof: It was pointed by a referee that the integrated solitary-pulse solution of Eq. (59) or (64) for the case $k_1 = 4$ can be expressed in a rather simple and compact form. It is

$$\dot{\theta} = \Omega_0(\zeta) = \frac{\sqrt{8}}{\tau}\frac{\cosh[-(\zeta - \zeta_0)/\tau]}{8 + \sinh^2[-(\zeta - \zeta_0)/\tau]}$$

or

$$\theta = \tan^{-1}\sqrt{8}\operatorname{csch}[-(\zeta - \zeta_0)/\tau].$$

The two maxima of $\dot{\theta}$ occur at $(\zeta - \zeta_0)/\tau = \pm\ln(\sqrt{6} + \sqrt{7}) = \pm 1.628307$.

REFERENCES

1. M. J. Konopnicki, P. D. Drummond, and J. H. Eberly, Opt. Commun. **36**, 313 (1981).
2. M. J. Konopnicki and J. H. Eberly, Phys. Rev. A **24**, 2567 (1981).

3. See L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975), Chap. 4; L. Matulic and J. H. Eberly, Phys. Rev. A **6**, 822, 1258E (1972).
4. R. E. Slusher and H. M. Gibbs, Phys. Rev. A **5**, 1964 (1972); **6**, 1255 (1972); N. Tan-no and Y. Higuchi, Phys. Rev. A **16**, 2181 (1977). The original work on one-photon self-induced transparency is in S. L. McCall and E. L. Hahn, Phys. Rev. **183**, 457 (1969).
5. F. T. Hioe, Phys. Rev. A **28**, 879 (1983).
6. F. T. Hioe, Phys. Rev. A **26**, 1466 (1982). It was not clearly stated that the Rabi frequencies were assumed to be real in this reference, in Ref. 5, and in some parts of Ref. 10 below.
7. F. T. Hioe, Phys. Rev. A **30**, 3097 (1984).
8. R. J. Cook and B. W. Shore, Phys. Rev. A **20**, 539 (1979).
9. F. T. Hioe, J. Opt. Soc. Am. B **4**, 1327 (1987).
10. F. T. Hioe, "Analytic solutions and dynamic symmetries in laser driven atomic excitation", in *Advances in Chemical Physics*, J. O. Hirschfelder, R. E. Wyatt, and R. D. Coalson, eds. (Wiley, New York, to be published).
11. See, e.g., A. R. Edmond, *Angular Momentum in Quantum Mechanics* (Princeton U. Press, Princeton, N.J., 1957), Sec. 3.7.
12. U. Fano and G. Racah, *Irreducible Tensorial Sets* (Academic, New York, 1959), p. 76.
13. B. R. Judd, *Operator Techniques in Atomic Spectroscopy* (McGraw-Hill, New York, 1963), p. 141.
14. A. Omont, Prog. Quantum Electron. **5**, 69 (1977).
15. Equation (15) was first given by S. Pancharatnam, Proc. R. Soc. London Ser. A **330**, 265 (1972), for a problem in magnetic resonance.
16. See, e.g., M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Addison-Wesley, Reading, Mass., 1962), p. 354.
17. This recurrence relation can be drived from Eq. (1.49) on p. 10 of M. Rotenberg, N. Metroplis, R. Bivins, and J. K. Wooten, Jr., *The 3-j and 6-j Symbols* (MIT Press, Cambridge, Massachusetts, 1959).
18. J. H. Eberly, Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627 (personal communication).
19. See, e.g., M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions* (Dover, New York, 1965), p. 771.
20. See Ref. 19, pp. 782 and 344.
21. I. N. Sneddon, *Special Functions of Mathematical Physics and Chemistry*, 2nd ed. (Oliver and Boyd, London, 1961), p. 54.
22. See Ref. 19, p. 776.
23. S. L. McCall and E. L. Hahn, Phys. Rev. **183**, 457 (1969).