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N-level quantum systems with Gell-Mann dynamic symmetry

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A set of conditions is presented for an N-level quantum system to possess the Gell-Mann-type dynamic symmetry, which was introduced in an earlier paper. The characteristic set of constants of motion that the system has when it possesses the symmetry and also the equivalent two-level system, which the system can be reduced to, are also presented.

Following the discovery by Gray *et al.*[1] and by Hioe and Eberly[2] of a set of constants of motion in a two-photon coherent excitation of a three-level system, the concept of dynamic symmetry in quantum electronics was explored further and clarified by the author[3–6] and others.[7–9] In particular, dynamic symmetry of the Gell-Mann type, which was first defined in Ref. 3 for a three-level system, was extended to N-level quantum systems in Ref. 4. Systems possessing the Gell-Mann dynamic symmetry were shown to have a characteristic set of constants of motion; this set resembles the set of quantum numbers associated with the isospin invariance, strangeness, charm, bottom and top, etc. in quark physics. In the group theoretical language, time-dependent Hamiltonians possessing the Gell-Mann symmetry simplify the dynamics by permitting the breakdown of SU(N) symmetry into its SU(2) and N-2 U(1) subgroups.

After the principal features of the Gell-Mann dynamic symmetry were identified, the questions turned to the conditions that an N-level quantum system must satisfy for it to possess the symmetry. Some examples of these conditions for the specific cases of N = 4–8 were given in Ref. 4. Although these examples were correct, some of the conditions in them were redundant and unneccesary. Indeed, it is somewhat unfortunate that the extra unneeded conditions (for $N > 4$) given in Ref. 4 might have given the impression that realization of the Gell-Mann dynamic symmetry is more complicated and less general than it actually is. It is my purpose in this paper to present a set of conditions that is simple, that can be stated generally for any value of N, and that probably permits the greatest number of arbitrary parameters in an N-level system for the system to possess the Gell-Mann dynamic symmetry.

We consider an N-level or N-state quantum system whose generally time-dependent Hamiltonian can be written in or reduced to the form given by

$$\hat{H}(t) = -\hbar \begin{bmatrix} 0 & \tilde{a}_{12}(t) & \tilde{a}_{13}(t) & \cdots & \tilde{a}_{1N}(t) \\ \tilde{a}_{21}(t) & \Delta_2(t) & \tilde{a}_{23}(t) & \cdots & \tilde{a}_{2N}(t) \\ \tilde{a}_{31}(t) & \tilde{a}_{32}(t) & \Delta_3(t) & \cdots & \tilde{a}_{3N}(t) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \tilde{a}_{N1}(t) & \tilde{a}_{N2}(t) & \tilde{a}_{N3}(t) & \cdots & \Delta_N(t) \end{bmatrix}, \quad (1)$$

where we assume that $\hat{H}(t)$ is Hermitian, and hence

$$\tilde{a}_{kj}(t) = \tilde{a}_{jk}{}^*(t) \quad (2)$$

for all j and k. The form of $\hat{H}(t)$ in Eq. (1) is so far general, and the zero in the first element of $\hat{H}(t)$ only expresses the fact that we can add to or subtract from $\hat{H}(t)$ any multiple of a unit matrix. Hamiltonians of such form appear in many dynamic problems—in laser physics as well as in particle collisions. We shall, as in previous work, use the language of laser physics and refer to the off-diagonal element $\tilde{a}_{jk}(t)$ as the half Rabi frequency associated with the transition from level j to k and the diagonal element $\Delta_k(t)$ as the cumulative detuning of $k - 1$ successive lasers from the corresponding sum of $k - 1$ level frequencies.

Let level 1 be the ground state, and let us label the other states or levels so that the dipole transition rule can be written as

$$\tilde{a}_{jk}(t) = 0 \qquad \text{if } |j - k| = \text{even number}. \quad (3)$$

We further assume that the Rabi frequencies for the allowed transitions have the form

$$\tilde{a}_{jk}(t) = \begin{cases} \tilde{a}_{jk} f(t) & \text{for } |j - k| = \text{odd number}, j \text{ odd}, \quad (4a) \\ \tilde{a}_{jk} f^*(t) & \text{for } |j - k| = \text{odd number}, j \text{ even}, \quad (4b) \end{cases}$$

where $\tilde{a}_{jk}$ are arbitrary constants and $f(t)$ is an arbitrary time-dependent function. In the simpler case, when $f(t)$ is a real function of time, Eq. (4) implies that all the Rabi frequencies have the same time dependence. In practice, if all the allowed transitions between the levels were derived from the same laser with a definite time-varying electric field envelope, then Eq. (4) would be satisfied automatically. When two or more lasers are used, Eq. (4) requires that they have the same time-dependent field envelope but that their amplitudes and frequencies can be arbitrary.

We shall refer to Eqs. (3) and (4) as conditions although they are actually automatically satisfied in a coherent excitation of an N-level system with a single laser. We do so in order to state the conditions as completely as possible so that the result will be equally applicable to many other problems in physics.

If the Hamiltonian elements satisfy the two conditions

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(i) $\Delta_n(t) = 0$ for all odd n (5a)

and

$\Delta_n(t) = \Delta(t)$,
an arbitrary function of time, for all even n;

(ii) $\tilde{a}_{jk} = a_j a_k{}^*$ all j, k (5b)
for $|j - k|$ = odd number, (6)

where a_j are arbitrary generally complex constants and $\tilde{a}_{jk}(t)$ satisfy conditions (3) and (4), we will show that the N-level quantum system possesses the Gell-Mann dynamic symmetry with a characteristic set of constants of motion, which we shall present. Physically, conditions (5) require the system to be at two-photon resonance for lasers tuned to any three successive levels and to have equal one-photon detunings at all times. This condition includes one-photon resonance, $\Delta(t) = 0$, at all times as a special case. Two-photon or one-photon resonance condition is often employed in practice. The type of dynamic symmetry that would result from this condition, without condition (6), will be discussed elsewhere.[10] For Gell-Mann dynamic symmetry to occur, condition (6), which we shall refer to as the product condition for the interaction parameters $\tilde{a}_{jk}$, is one that gives the system its distinctive feature and is probably the simplest condition that can be stated for an N-level system to possess the Gell-Mann symmetry.

Equation (6) implies that although the number of interaction parameters $\tilde{a}_{jk}$ in the N-level system is $(N^2 - 1)/2$ if N is odd and is $N^2/2$ if N is even, assuming that $\tilde{a}_{jk}$ are all complex, the number of *independent* interaction parameters is only $2(N-1)$. If $\tilde{a}_{jk}$ are all real, then we divide all the counts by 2. We may assume, for convenience, that the independent interaction parameters are $\tilde{a}_{12}, \tilde{a}_{23}, \ldots, \tilde{a}_{N-1,N}$ and their complex conjugates; the remaining interaction parameters then must always be expressible in terms of these independent interaction parameters, according to Eq. (6), and hence they can no longer be arbitrary if the system is to possess the Gell-Mann dynamic symmetry. For $N = 3$, Eq. (6) is no condition at all because the number of interaction parameters and the number of independent interaction parameters permitted by Eq. (6) are both equal to 4. Thus the two-photon resonance condition in a three-level system is the only condition needed for the system to possess the GGell-Mann symmetry for any Rabi frequencies so long as they have the same time dependence as required by Eq. (4). On the other hand, for $N \geq 4$, Eq. (6) limits the number of arbitrary interaction parameters to $2(N-1)$ or to $N-1$ if all the arbitrary interaction parameters are real. The common time-dependent function $f(t)$ is still arbitrary.

More explicitly, an N-level system whose Hamiltonian can be expressed in the following form possesses the Gell-Mann dynamic symmetry:

$$\hat{H}(t) = -\hbar \begin{bmatrix} 0 & a_1a_2{}^*f(t) & 0 & a_1a_4{}^*f(t) & 0 & \cdots \\ a_2a_1{}^*f^*(t) & \Delta(t) & a_2a_3{}^*f^*(t) & 0 & a_2a_5{}^*f^*(t) & \\ 0 & a_3a_2{}^*f(t) & 0 & a_3a_4{}^*f(t) & 0 & \\ a_4a_1{}^*f^*(t) & 0 & a_4a_3{}^*f^*(t) & \Delta(t) & a_4a_5{}^*f^*(t) & \\ \vdots & & & & & \end{bmatrix}, \quad (7)$$

where $a_1, a_2, \ldots$ are arbitrary constants and $f(t)$ and $\Delta(t)$ are arbitrary time-dependent functions.

We now show that conditions (3)–(6) are sufficient for the system to possess Gell-Mann dynamic symmetry. Following Ref. 4, an N-level system with the Hamiltonian $\hat{H}(t)$ is said to possess the Gell-Mann dynamic symmetry if its Hamiltonian $\hat{H}(t)$ can be unitarily transformed into the form given by

$$\hat{\mathcal{H}}(t) = -\hbar \begin{bmatrix} \begin{matrix} 0 & h_{12}(t) \\ h_{21}(t) & \Delta(t) \end{matrix} & 0 & 0 & \cdots \\ 0 & \begin{matrix} 0 & 0 \\ 0 & \Delta(t) \end{matrix} & 0 & \\ 0 & 0 & \begin{matrix} 0 & 0 \\ 0 & \Delta(t) \end{matrix} & \\ \vdots & & & \end{bmatrix}, \quad (8)$$

where $\hat{\mathcal{H}}(t)$ is obtained from $\hat{H}(t)$ by

$$\hat{\mathcal{H}}(t) = \hat{U}^{\dagger}\hat{H}(t)\hat{U}, \quad (9)$$

where $\hat{U}$ is a time-dependent unitary matrix.

We shall construct the appropriate unitary matrix $\hat{U}$ column by column as follows. The first two columns are the vectors

$$\mathbf{u}_1 = \begin{bmatrix} M_1{}^{-1}a_1 \\ 0 \\ M_1{}^{-1}a_3 \\ 0 \\ M_1{}^{-1}a_5 \\ 0 \\ M_1{}^{-1}a_7 \\ 0 \\ \vdots \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ M_2{}^{-1}a_2 \\ 0 \\ M_2{}^{-1}a_4 \\ 0 \\ M_2{}^{-1}a_6 \\ 0 \\ M_2{}^{-1}a_8 \\ \vdots \end{bmatrix}, \quad (10)$$

where M_1 and M_2 are normalization factors given by

$$M_1 = (|a_1|^2 + |a_3|^2 + |a_5|^2 + |a_7|^2 + \ldots)^{1/2}, \quad (11a)$$

$$M_2 = (|a_2|^2 + |a_4|^2 + |a_6|^2 + |a_8|^2 + \ldots)^{1/2}. \quad (11b)$$

For the remaining columns, we first consider the set of vectors $\mathbf{v}_n$ for $n = 3, 4, \ldots, N$ given by

$$\mathbf{v}_3 = \begin{bmatrix} M_3^{-1}a_3^* \\ 0 \\ -M_3^{-1}a_1^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ M_4^{-1}a_4^* \\ 0 \\ -M_4^{-1}a_2^* \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} M_5^{-1}a_5^* \\ 0 \\ 0 \\ 0 \\ -M_5^{-1}a_1^* \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix},$$

$$\mathbf{v}_6 = \begin{bmatrix} 0 \\ M_6^{-1}a_6^* \\ 0 \\ 0 \\ 0 \\ -M_6^{-1}a_2^* \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad \mathbf{v}_7 = \begin{bmatrix} M_7^{-1}a_7^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -M_7^{-1}a_1^* \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad \mathbf{v}_8 = \begin{bmatrix} 0 \\ M_8^{-1}a_8^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -M_8^{-1}a_2^* \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \tag{12}$$

and so on. Each vector has only two non-zero elements, the general pattern by which they appear can be deduced from the examples given above and here for $n > 2$:

$$M_n = \begin{cases} (|a_1|^2 + |a_n|^2)^{1/2} & \text{if } n \text{ is odd}, & (13a) \\ (|a_2|^2 + |a_n|^2)^{1/2} & \text{if } n \text{ is even}. & (13b) \end{cases}$$

The vectors $\mathbf{v}_n$ for $n = 3 - N$ can be verified to be the eigenvectors of $\hat{H}(t)$ given by Eq. (7) when we set $\Delta(t) = 0$, corresponding to the zero eigenvalue. Let $\mathbf{v}_n^\dagger$ denote the row vectors that are the complex-conjugate transpose of $\mathbf{v}_n$. The $\mathbf{v}_n$ vectors are not orthogonal, but we can construct, by superposition, an orthonormal set $\mathbf{u}_n$, $n = 3, 4, \ldots, N$ by the familiar Gram–Schmidt orthogonalization process as follows. We set

$$\begin{aligned} \mathbf{u}_3 &= \mathbf{v}_3, \\ \mathbf{u}_4 &= \mathbf{v}_4, \\ \mathbf{u}_5 &= \mathbf{u}_3 + \beta_{55}\mathbf{v}_5, \\ \mathbf{u}_6 &= \mathbf{u}_4 + \beta_{66}\mathbf{v}_6, \\ \mathbf{u}_7 &= \mathbf{u}_3 + \beta_{75}\mathbf{u}_5 + \beta_{77}\mathbf{v}_7, \\ \mathbf{u}_8 &= \mathbf{u}_4 + \beta_{86}\mathbf{u}_6 + \beta_{88}\mathbf{v}_8, \end{aligned} \tag{14}$$

and so on, where we choose β_{55} and β_{66} and suitable normalization constants so that

$$\mathbf{u}_3^\dagger \cdot \mathbf{u}_5 = \mathbf{u}_4^\dagger \cdot \mathbf{u}_6 = 0,$$
$$\mathbf{u}_5^\dagger \cdot \mathbf{u}_5 = \mathbf{u}_6^\dagger \cdot \mathbf{u}_6 = 1$$

and then choose β_{75}, β_{77}, β_{86}, and β_{88} and suitable normalization constants so that

$$\mathbf{u}_3^\dagger \cdot \mathbf{u}_7 = \mathbf{u}_5^\dagger \cdot \mathbf{u}_7 = \mathbf{u}_4^\dagger \cdot \mathbf{u}_8 = \mathbf{u}_6^\dagger \cdot \mathbf{u}_8 = 0,$$
$$\mathbf{u}_7^\dagger \cdot \mathbf{u}_7 = \mathbf{u}_8^\dagger \cdot \mathbf{u}_8 = 1,$$

and so on. The $N - 2$ column vectors $\mathbf{u}_n$ for $n = 3, \ldots N$ so constructed make up the remaining $N - 2$ columns of the unitary matrix $\hat{U}$, which we desire, and its complex-conjugate transpose makes up the matrix $\hat{U}^\dagger$. We can verify that $\hat{U}$ transforms $\hat{H}(t)$ of Eq. (7) into $\hat{\mathcal{H}}(t)$ of Eq. (8). The explicit expressions for the $\mathbf{u}_n$ for $n = 3 - N$ are not required for this verification purpose; all that is needed is to note the simple result when $\mathbf{u}_n$ is multiplied from the left by $\hat{H}(t)$ of Eq. (7) and that $\mathbf{u}_m^\dagger \cdot \mathbf{u}_n = \delta_{mn}$.

We also find that in Eq. (8)

$$h_{12}(t) = M_1 M_2 f(t), \tag{15a}$$
$$h_{21}(t) = M_1 M_2 f^*(t) = h_{12}^*(t), \tag{15b}$$

where M_1 and M_2 are given by Eq. (11) and $M_1M_2 = \sum\sum_{j<k}|\tilde{a}_{jk}|^2)^{1/2}$ are given by using Eqs. (3), (4), and (6). We have thus shown that an N-level system whose Hamiltonian $\hat{H}(t)$ satisfies conditions (3)–(6), or, more explicitly, is given by Eq. (7), possesses the Gell-Mann dynamic symmetry and is reducible to a two-level system for which a large number of analytic solutions are known.[11] The unitary matrix $\hat{U}$ that we have constructed here is identical to those matrices constructed in Ref. 4 for $N = 3$ and $N = 4$ (except for the generalization to the complex values of a_j); however, $\hat{U}$ differs from those constructed in Ref. 4 when $N \geq 5$ and is better because, in effect, it removes some of the conditions placed on the interaction parameters there for these cases. The only conditions required here for the interaction parameters are those given by conditions (3), (4), and (6).

Let $\psi(t)$ be the solution of the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}(t)\psi. \tag{16}$$

Multiplying this equation from the left by $\hat{U}^\dagger$, we get

$$i\hbar \frac{\partial \psi'}{\partial t} = \hat{\mathcal{H}}(t)\psi', \tag{17}$$

where $\hat{\mathcal{H}}(t)$ is given by Eq. (8) and

$$\psi' = \hat{U}^\dagger \psi. \tag{18}$$

If $\hat{H}(t)$ possesses the Gell-Mann symmetry, then the Schrödinger equation for the equivalent two-level system is

$$i\hbar \frac{\partial}{\partial t}\begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix} = -\hbar \begin{bmatrix} 0 & h_{12}(t) \\ h_{21}(t) & \Delta(t) \end{bmatrix} \begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix}, \tag{19}$$

where $h_{12}(t)$ and $h_{21}(t)$ are given by Eqs. (15). It follows from Eqs. (8), (17), and (18) that we have the following set of constants of motion:

$$|\mathbf{u}_1^\dagger \cdot \psi(t)|^2 + |\mathbf{u}_2^\dagger \cdot \psi(t)|^2 = \text{const.}, \tag{20a}$$
$$|\mathbf{u}_n^\dagger \cdot \psi(t)| = \text{const.}, \qquad n = 3, 4, \ldots, N, \tag{20b}$$

which are characteristic of the Gell-Mann symmetry. The a_j's appearing in the constants of motion can be replaced by the physical interaction parameters $\tilde{a}_{jk}$ appearing in the Hamiltonian [Eq. (7)] by using Eq. (6). For the three-level system, Eq. (20b) can be verified to be the same as the constant of motion first discovered by Gray *et al.*[1] and by Hioe[2,3] and Eberly.[2] When $N = 4$, Eqs. (20) can be shown to be equivalent to the set of constants of motion given in Ref. 4 in terms of the density matrix elements, and when $N > 4$, the set of constants of motion given by Eqs. (20) is new.

Equation (20a) expresses the conservation of the total level population in the equivalent two-level system to which the N-level system possessing the Gell-Mann symmetry was reduced. The constants of motion given by Eq. (20b) imply coherent population trapping in the sense that the population initially in the linear combination $|\mathbf{u}_n{}^\dagger \cdot \psi(0)|$ remains there.

In summary, we have presented a set of conditions [Eqs. (3)–(6)] for the Hamiltonian of an N-level system [which is shown more explicitly in Eq. (7)] for the system to possess the Gell-Mann dynamic symmetry. These conditions permit the Hamiltonian to have $2(N - 1)$ arbitrary interaction parameters with the same but arbitrary time dependence and also permit an arbitrary detuning function. The consequence of this symmetry is that the system is shown to have a characteristic set of constants of motion given by Eqs. (20) and that the system is mathematically reducible to an equivalent two-level system whose dynamic evolution is given by Eq. (19). It is interesting to compare the N-level system having the Gell-Mann symmetry with an N-level Cook–Shore[12] system or its generalization, which is an N-level system having the SU(2) symmetry,[6] and also with an N-level system having the type of symmetry discussed in Ref. 10. These and a few other cases make up a special group of N-level systems that can be analyzed analytically and that exhibit many unexpected and unusually interesting sets of properties.

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