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Matching Functions and Graphs at Multiple levels of Bloom's Revised Taxonomy

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ABSTRACT: This paper illustrates the power of Bloom's revised taxonomy for teaching, learning and assessing [3] in aligning our curriculum expectations and our assessment tools in multivariable calculus. The particular assessment tool considered involves a common matching problem to evaluate students' abilities to think about functions from graphical and formulaic representations. Through this analysis we gain additional understanding of why students may have difficulty in performing well on certain activities.
KEYWORDS: Bloom's taxonomy, curriculum alignment, problem analysis, calculus (multivariable)

1 Introduction

After a little more than a decade of college- and university-level teaching, I have taught vector calculus, aka multivariable calculus, aka calculus III, almost a dozen times, with good teaching evaluations and consistently good student acheivement. So, one might reasonably suppose that I have some idea about what I am doing. Alas, each and every semester, this seems to be proven wrong. This experience, among others, has led to a radical reformulation of my philosophy of teaching as a four stage process. First, I make assumptions about my students. In principle, these assumptions are based on things like their abilities and classes they've (supposedly) taken, and my expectations for the course. After this "Assumption-Making" stage, I plan my instruction, design lectures and classroom activities, assign and monitor projects, and give and grade tests in the "Teaching" stage. This stage is followed by what I will refer to optimistically as the "Discovery" stage where I attempt to discern why almost all of my earlier assumptions about the students wwere completely wrong. Then I cap it all off with the final stage, "Amnesia", where I forget all of these lessons and repeat similar mistakes in the future. The following is a tale about my experiences with a seemingly simple assignment from the beginning of multivariable calculus. I will share my experiences during the first three stages of instruction, in the hopes that some of us will not proceed to the fourth stage with respect to this material. Throughout, I will make use of the taxonomy for teaching, learning, and assessing published in 2001 [1] to provide a theoretical framework for the "Discovery" stage of this experience. This taxonomy will explain why three superficially similar tasks involving matching graphs of functions to their algebraic representations are not in fact similar at all. This lack of similarity is such that, in my experience to date, students who perform well on one or two of these tasks often do much worse on the final task. Since tasks such as these are commonly used as assessment tools from pre-calculus to multivariable calculus in order to provide experience with the multiple representations of objects, this lack of similarity is indeed of critical importance.

2 Overview of the Taxonomy

Our primary tool for understanding the differences in these problems will be the 2001 revision of Bloom's original taxonomy found in [1]. Half a century after its introduction, Bloom's taxonomy [2] is well-known to many educators. Its categorization of cognitive tasks helps to focus teaching and balance instructional objectives (See Table 1). This tool has had far-reaching impacts on every level of education, especially with regard to teaching higher order thinking skills. Yet, many educators seem completely unaware of the 2001 revision of this taxonomy, a revision led by one of Bloom's original co-authors.

The revised taxonomy, illustrated in Table 2, involves two dimensions, rather than the one cognitive dimension of the original framework. One of these is the knowledge domain, which references the type of knowledge the task uses: factual, conceptual, procedural, or metacognitive. The second dimension is a slight reorganization of the six levels of the original cognitive domain. The new cognitive dimension includes, in order of increasing cognitive depth: remember, understand, apply, analyze, evaluate, and create. In the original taxonomy, the highest levels appear in reverse order; the new taxonomy also presents each cognitive activity as a verb, indicating the action that a learner is demonstrating. The new taxonomy is then the Cartesian product of these two sets, resulting in twenty-four types of instructional goals, activities, and assessments. The revision was introduced in a volume emphasizing the need to align our educational objectives, in-

Knowledge
Comprehension
Application
Analysis
Synthesis
Evaluation

Table 1: The original taxonomy of educational objectives, presented in increasing difficulty from top to bottom.

	Factual	Conceptual	Procedural	Metacognitive
Remember				
Understand				
Apply				
Analyze				
Evaluate				
Create				

Table 2: The revised taxonomy matrix showing the cognitive dimensions vertically and the knowledge domains horizontally.

structional tasks, and assessment activities. In the author's experience, the framework provided by this model is extremely useful for accomplishing this alignment. For a more complete overview of the revised taxonomy and a comparison with the original taxonomy see [3].

The value of the taxonomy is in providing us with a way to look not only at the cognitive depth of an activity, but also at how those depths interact with different types of knowledge. For example, we frequently make use of the levels: remember factual knowledge, understand conceptual knowledge, and apply procedural knowledge. Our instructional objectives and student learning goals frequently contain language that unambiguously refers to these three types of tasks. However, the three similar-looking assessment activities discussed below each appear at a different place in the taxonomy, emphasizing that even though tasks may look similar and use similar language, they can require quite diverse modes of thinking. I can only wish that I had realized this long before I did, as years of teaching multivariable calculus and using what is referred to below as Problem A would have proven much more rewarding, I suspect.

3 Three Problems

In teaching the course, I usually use the multivariable calculus text from the Calculus Consortium [4]. In the early chapters of the text, there are a wide variety of problems designed to help the students begin to visualize functions of several variables making use of surface plots, sections, and contours. There is one problem, in particular, that I invariably assign; the problem requires the students to match the equations of nine different functions of two variables with nine different surface plots. Despite various approaches used in the course (using MAPLE, lots of in class discussion to clarify and illustrate thinking, and other similar strategies) this is often the question eliciting the most questions from students and the lowest overall performance. The reason for this difficulty points to the subtlety with which different types of thinking can be hidden inside a problem and has, I think, interesting consequences for the rest of the mathematics curriculum, with particular application to the calculus sequence.

The problem to which I refer appears as number 16 in section 12.2 (pp. 573-4) of the third edition of the text [4]. It is reproduced here in in modified form as Figure 1, and will be referred to as Problem A in the remainder of the discussion. The problem provides two lists: A list of functions of the form $z = f(x, y)$ with two independent variables and a list of surface plots to be matched to these functions. The problem states that it should be done without a graphics package. Indeed, when students attempt to use MAPLE, they discover a variety of problems, among them the lack of scale on the axes in the graphs. Without this cue, many of them cannot find a suitable representation of the function on screen to match the graphs shown. Further, at this point in the course, students have only been shown how to graph surfaces on a rectangular domain; many of the graphs in the set have circular symmetry and are shown graphed on a suitable circular domain, making matching between the computer screen and the printed page more difficult.

Past experience has shown that students often resort to wild guessing. In discussing the problem with students, I emphasize that the problem should be solved using logic to deduce the connections. The symmetry of the functions should be considered; the behavior near the origin and at infinity should be accounted for. One can also narrow down the list of suspect functions to match with a given graph by considering the range of the functions, since some have strictly positive ranges, while others are positive in some quadrants and not others, and some take values from almost anywhere. This method of reasoning through the problem, a method that comes naturally to many of us as mathematicians, seems new to many students in spite of having worked with similar-appearing problems prior

(a) $z = xye^{-(x^2+y^2)}$	(b) $z = \cos\left(\sqrt{x^2+y^2}\right)$	(c) $z = \sin y$
(d) $z = -\frac{1}{x^2+y^2}$	(e) $z = \cos^2 x \cos^2 y$	(f) $z = \frac{\sin\left(x^2+y^2\right)}{x^2+y^2}$
(g) $z = \cos(xy)$	(h) $z = \left(2x^2+y^2\right) e^{1-x^2-y^2}$	(i) $z = \lvert x \rvert \lvert y \rvert$
(I)	(II)	(III)
(IV)	(V)	(VI)
(VII)	(VIII)	(IX)

Figure 1: Problem A, the matching problem modified from [4] number 16, pp. 573-4.

to this in pre-calculus and calculus courses. As we will see, the taxonomy will illustrate one reason for this difficulty is rooted in a misalignment of instructional goals with assessment tools.

Some may react to the last statements with incredulity. After all, the reform movement in calculus has emphasized problems of this sort, with graphs and equations to match as practice and assessments for dealing with multiple representations of functions, and many of these students have been through such courses. The reform approach has made it "down" into the pre-calculus curriculum, providing students with more opportunities to encounter such problems and make their reasoning explicit. While this is certainly true, the revised taxonomy makes it clear that these earlier problems may contribute to student difficulty with problem A.

Simply put, the other problems involving the matching of graphs and equations are only similar to Problem A on the surface. To see why this is true, we will consider two different matching problems that might appear in a pre-calculus course. Problem B (see Figure 2) provides the students with a list of single variable functions and graphs to be matched. The functions are taken from all over the pre-calculus curriculum: exponentials, sines, polynomials and power functions. In Problem C (Figure 3), the variety of function families represented is limited to one family. There may be nine trigonometric or quadratic functions represented, along with their graphs. These three problem types, while taking similar forms, involve three different solution strategies, and indeed, occupy three different places in the revised version of Bloom's well-known taxonomy.

(a) $y = e^x$	(b) $y = \sin x$	(c) $y = \cos x$
(d) $y = x^2$	(e) $y = \frac{1}{x}$	(f) $y = \tan x$
(g) $y = \sqrt{x}$	(h) $y = \|x\|$	(i) $y = e^{-x^2}$
(I)	(II)	(III)
(IV)	(V)	(VI)
(VII)	(VIII)	(IX)

Figure 2: Problem B, matching functions to graphs across many function families.

(a) $y = x^2$	(b) $y = \frac{1}{2}x^2$	(c) $y = 2x^2$
(d) $y = x^2 + 1$	(e) $y = x^2 - 1$	(f) $y = (x-2)^2$
(g) $y = (x+3)^2$	(h) $y = (x-1)^2 + 2$	(i) $y = (x-2)^2 + 1$
(I)	(II)	(III)
(IV)	(V)	(VI)
(VII)	(VIII)	(IX)

Figure 3: Problem C, matching functions to graphs within a single function family. Each graph is displayed $[-3, 3] \times [-3, 10]$.

4 Solving Problems B and C

How would students approach problem B? Most students seem to proceed by matching types of functions. Typical reasoning would include statements like: "Trig functions oscillate, so function (b) goes with either graph II, IV or VI." "Exponentials like (a) are always increasing and concave up, so it must be graph VII." But what are the students really doing in this process? Are they reasoning through the problem and connecting the graphs with the functions because of the behaviors encountered? Not really. These students have all studied the various pre-calculus families of functions extensively. They know what the graphs are supposed to look like. They are, in essence, performing a parallel processing pattern recognition task. Thus, the students are involved simply in *recalling factual knowledge*. Occasionally, there will be two functions from a given family. Deciding which is to be matched with a particular graph, students may resort to solution techniques similar to those used in solving problem C, but they will rarely approach the depth of thought required to solve problem A.

Problem C requires a little more ingenuity than problem B. If all the functions are taken from the same (or similar) function families, then one cannot simply use pattern recognition. In the example shown in Figure 3 the family of functions is the family of quadratic functions. To match these graphs and equations, students may need to calculate the zeros of each function or locate the vertex of the parabola and then compare these to the graphs, searching for matches. This type of solution technique makes use of a different level of the taxonomy, slightly higher up the cognitive domain than the tools used in problem B. For the most part, this is a matter of *applying procedural knowledge* to the situation: students must apply their knowledge of the properties of trigonometric functions to the formulas, and then apply this knowledge to read the properties from the graphs. In some ways, students must also *understand conceptual knowledge* in order to discern the features of the graphs that are important to help them evaluate the options. While both of these levels of reasoning are higher in the cognitive domain than Problem B, we will see that Problem A is yet another order of thinking above this.

5 Solving Problem A

Problem A is distinct from both B and C. One distinction is in the nature of the functions encountered. Both B and C make use of functions the students have studied extensively, allowing them a catalog from which to match features - either by recognition or by calculation. Problem A, on the other hand, is made up of pieces of familiar functions, but none of these functions would ever be studied as a "function family" worthy of general knowledge. In fact, at the point in the text where this problem occurs, students are being introduced to functions of several variables for the first time. They are encouraged to think through the problem rather than to simply recall previously divulged information, so the students have no catalog of functions from which to draw examples. While the process of calculating some of the features of the function to match with the graph has some similarities to the application of knowledge used in solving problem C, there is a major difference. Since the functions in problem C are previously studied quantities, the students are aware of exactly which calculations reveal exactly the set of features that will distinguish one function from another. By contrast, students are in open water for problem A, with a host of possible features to explore, and no certainty that any of these features will reveal the secrets. They are forced to rely on much more general properties of functions - symmetry, range, limiting behaviors, slope, concavity, and the presence of oscillations, among other possible revealing features.

Having students share their thinking out loud in discussing Problem A in class reveals many interesting insights. In fact, one rarely hears two students in the room (who did not work together on the assignment) use identical or even similar methods to solve the problem. This is one of the reasons I use such a problem: to illustrate the variety of approaches and to help break students out of the "one problem implies one method implies one answer" mentality. A complete analysis of the problem shows that there is no single pattern of thought that can be easily replicated from the study of one function to the study of the others in the list. The difficulties are even greater when one considers the representations of the functions graphically; since these are static snapshots of a three-dimensional object graphed on a set domain, some of the important features may be almost impossible to locate on the graph either because it is hidden behind the visible surface or because it is simply not graphed.

This problem then, incorporates several levels of the taxonomy at once. Students must *understand conceptual knowledge* in order to even think about the symmetry or asymptotic behavior of the functions or their graphs. They must *analyze factual knowledge* presented in the graphs in order to determine the properties of the functions displayed. They must *evaluate procedural knowledge* in order even come up with a strategy for approaching this problem and grouping the graphs or functions into similar categories. In some cases they must even *apply procedural knowledge* to compute specific details about the functions, such as their value at the origin. Further, they must make sense out of their conceptual knowledge of what symmetry

	Factual	Conceptual	Procedural	Metacognitive
Remember	B			
Understand		A		
Apply			A, C	
Analyze	A			
Evaluate			A	
Create				

Table 3: Comparison of problems A, B, and C on the taxonomy illustrates the differences in cognitive depth and knowledge types used in solving each problem.

is and how the pieces of the functions involved behave. Thus, problem A requires multiple modes of thinking, in sharp contrast to problems B and C which focus on a single mode of thinking. These modes are summarized in Table 3.

We cannot expect a single experience in one course, no matter how well designed, to prepare students for what is almost a completely different way of thinking about a problem. We must explore opportunities for such thinking in all mathematics courses. In fact, for many of us, this type of thinking is the hallmark of mathematical or analytical reasoning, so it seems that this is a more important message for students to take from our courses than specific content regarding the appearance of multi-variable functions. To begin with, we could easily include such examples in earlier courses, say pre-calculus. Even though students are only beginning to explore the different representations of function, and indeed the concept of function itself, once students have some basic catalog of functions, a collection involving composition, addition, or multiplication of these functions could be presented both algebraically and graphically with students expected to match the two representations through application of their prior knowledge and tools. They could determine the zeros for some of these to aid in their analysis, identify the y-intercept, or plot a few points. The main point to remember here is that we are not assessing what students know about the particular functions; we are assessing how well they understand the components of different representations of functions and how well they can apply their collection of analytical tools to the problem.

6 Implications for Teaching

One thing certainly stood out from all this: I need to more closely align my course objectives, classroom activities, and assessments. This does not mean that I should "teach to the assessment." Rather, it means that I need to provide appropriate experiences in my course goals and instructional activities that make use of the same cognitive and knowledge dimensions as the assessments I use. But no matter how much I think I am modelling my thought process, no matter how many similar types of activities the students encounter in my course, no matter how frequently I expect them to use these concepts (such as symmetry and asymptotic behavior) to analyze a function, they still seem to fall short of my expectations on Problem A. This could mean that my expectations are too high, even after "training them on similar tasks." Or it could mean that students have met so many superficially similar problems in the past that they expect to be able to continue using the old strategies from Problem B or C. How can we get them past this so that they do not give up and guess wildly?

Thus, it is possible that by using superficially similar tasks, we have primed the students for some improper reasoning. This would not be the first time for such an occurrence. Research on students in middle [6] and secondary school [7] suggest that the emphasis placed on proportional reasoning leads students to misapply this reasoning in situations, such as area and volume, where it is not appropriate. In some cases, they apply the logic in spite of common sense knowledge that contradicts their answers. This is yet another example of how students learn something different from what we think we are teaching. In the case of problem A, however, students expect to apply the same types of reasoning as they did in the past on similar problems (B and C). But without the appropriate facts in memory, their process is blocked.

One solution to help students would be to change the statement of the problem, perhaps by having them narrow down the possible matches for each function and then working within these subsets explicitly. For example, the directions for the problem could require students to first make several lists of possible matches for each function in the problem. They could list all the possible matches based on symmetry, all possible matches based on range, and so forth. Then they could be asked to select the graphs appearing on all the lists, and further eliminate those with features that are incompatible. This type of modification to the problem would amount to adding some of the instructional ideas into the problem itself and demonstrate important problem solving techniques. However, it is my experience that students tend to view sub-tasks to a problem as separate problems, rather than steps toward achieving an overall solution. This often prevents them from seeing how the individual steps fit together into a single approach to the overall problem. Further, based on the checklist of items for designing matching problems given in [5], this would seriously violate the tenet for clear and concise directions (p. 104).

There are certainly other explanations for student difficulty with such matching problems - after all, visualization in three dimensions is not easy for students, especially after spending years focusing on two-dimensional thinking. Additionally, according to [5], selected response assessment items, which includes matching questions, are excellent choices for evaluating "mastery of facts, concepts and even generalizations" (p. 70) but cannot evaluate all aspects of reasoning proficiency. It is thus reasonable to ask whether the format itself, and students' prior experiences with the format, are a significant part of the problem. Stiggins [5] (pp. 84-5, 104-5) also provides a list of other considerations in developing matching questions. However, these issues are almost all adequately addressed in the design of the original version of problem A. It seems then, that the only question in this regard is whether there is a match between the goals being evaluated and the question format. But unless one simply asks the students to graph the equations or asks them to propose reasonable equations for the graphs shown, it would be difficult to get at the same depths of reasoning as Problem A is intended to elicit.

Regardless of other possible explanations for student difficulties, we can see that the taxonomy provides a powerful tool for checking whether the problems we use are aligned with our teaching techniques and our goals. Of course, one goal of almost all mathematics faculty that I have spoken with is that students should work with unfamiliar problems. But to do this, we have to make sure we have provided the skills, concepts, and facts they need as well as the thinking process to analyze the problem and the metacognitive skills they need to monitor their own progress. Most of us would probably not give a completely novel problem to the students in our class and grade them only on whether or not they successfully solved the problem; we would also assess them on how well they demonstrate and application of the thinking processes and concepts of the course to the analysis of the problem.

It is my hope that this comparison of the three problems has served to clarify three of the four aspects of the teaching "model" mentioned in the introduction. By carefully considering the kinds of problems and thinking with which students are familar, we should be able to move our assumptions about students closer to reality in the "Assumption-Making" stage. By considering the goals of the course and carefully aligning our instructional activities and assessments with those goals, using the taxonomy discussed above, the "Teaching" stage should be more rewarding by letting students demonstrate the growth and learning we expect of them. The taxonomy also provides a valuable tool for use in the "Discovery" stage, where we can look for alignment, identify gaps, and improve our instructional planning process. Finally, if all goes well, this will help us forget about the fourth stage altogether.

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7 Biographical Sketch

Kris Green is an associate professor in the Mathematical and Computing Sciences department at St. John Fisher College. He is a national Project NExT fellow (1999, Brown Dot) with a Ph.D. in Applied Mathematics, although most of his friends believe that it is actually a Ph.D. in Star Wars Studies. Lately, his interests have been focused on mathematics and science education at all levels. When not engaged in scholarly activities, he enjoys taking walks with his wife, practicing Isshinryu karate and eating at Taco Bell.