Wave Equation on Spherically Symmetric Lorentzian Metrics

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Abstract
Wave equation on a general spherically symmetric spacetime metric is constructed. Noether symmetries of the equation in terms of explicit functions of θ and ϕ are derived subject to certain differential constraints. By restricting the metric to flat Friedman case the Noether symmetries of the wave equation are presented. Invertible transformations are constructed from a specific subalgebra of these Noether symmetries to convert the wave equation with variable coefficients to the one with constant coefficients.

Keywords
Partial-Differential Equations, Conservation-Laws, Noether, Separation, Variables

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Wave equation on spherically symmetric Lorentzian metrics

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I. INTRODUCTION

As for as Lie point symmetries[1,19,27,29] are concerned, they have important relationship between them and the conservation laws admitted by partial differential equations (pdes). It is for this that finding conservation laws associated with symmetries has been a topic of interest for many researchers (see, e.g., Refs. 6–18, 20, 24–26, 28 and 30). A systematic procedure of determining conservation laws (associated with variational symmetries) for systems of Euler-Lagrange equations is indeed the famous Noether theorem.[15] Direct construction methods for multipliers and hence the conservation laws,[32] Lagrangian approach for evolution equations[23] and formula for relationship between symmetries and conservation laws, irrespective of the existence of a Lagrangian of the system[8] have been investigated.

The nonlinear $(1+1)$ wave equation

$$u_{tt} - \frac{\partial}{\partial x}(f(u)u_x) = 0, \tag{1.1}$$

describing waves in one dimension involving arbitrary velocity function arises when transmitting a signal on a transmission line with material properties that are changing along the line. Ames *et al.*[33] obtained a complete group classification for its admitted point symmetries with respect to the wave speed function $f(u)$ and constructed explicit invariant solutions in some particular cases. Extending their work, Bokhari *et. al.* studied conservation laws of the nonlinear $(n+1)$ wave equation

$$u_{tt} - \mathbf{div}(f(u)\mathbf{grad}\, u) = 0, \tag{1.2}$$

involving an arbitrary function of the dependent variable.[34] Since a full Lagrangian of this equation does not exist, a partial Lagrangian approach was used to find partial Noether operators for various choices of $f(u)$ and a relationship between the partial Noether symmetry operators and the Lie symmetries of the equation was obtained.

Generally, nonlinearity in the pdes is introduced due to some "*ad hoc*" assumptions. An account of such work is available in Refs. 20–22. For example, symmetry classification for a number of wave equations in flat space is discussed in Ref. 5 and 33. In the present paper, we extend the earlier

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investigations[5] by studying Noether symmetries of wave equation by coupling it with spherically symmetric spacetime metric.[31] Since the spacetime metric is an explicit function of θ and ϕ, the equations determining Noether symmetries for this metric are solved up to explicit functions of these coordinates. In order to relate the problem to a realistic physical situation, we solve the equations in Friedman models representing exact expanding solutions of the Einstein field equations.[31] The plan of the paper is as follows: In Sec. II, we discuss Noether symmetries of a (3 + 1) wave equation on a spacetime metric admitting spherical symmetry. In Sec. III, we solve the wave equation to find Noether symmetries admitted by it and using a conformal transformation re-cast it into a constant coefficient equation. A brief discussion of results is given in the last section.

II. NOETHER SYMMETRIES OF A (3 + 1) WAVE EQUATION ON SPHERICALLY SYMMETRIC LORENTZIAN METRICS

We begin by briefly defining Noether symmetries. Noether symmetries are associated with the differential equations possessing Lagrangians. These symmetries describe physical features of differential equations in terms of conservation laws they admit. Emmy Noether in 1918, proved a remarkable result that for systems arising from a variational principle, every conservation law of the system comes from a corresponding symmetry property. For example, invariance of a variational principle under time translation gives rise to the conservation of energy for the solutions of the associated Euler-Lagrange equations and invariance under a spatial translations implies conservation of linear momentum. This basic principle constitutes the first fundamental result in the study of classical systems with prescribed symmetries. Noether's method is a systematic procedure for constructing conservation laws for complicated systems of pdes[2] These conservation laws are known as Noether symmetries and leave the action integral invariant.[3] Noether proved[4] that for every infinitesimal transformation admitted by the action integral of a Lagrangian system a conservation law exists. Noether's theorem provides an algorithm to construct conservation law for any admitted infinitesimal transformation. The importance of this result is that one can find all such transformations by examining the invariance properties of the Euler Lagrange equations.[17] Given a Lagrangian $L(q^i, \dot{q}^i; t)$ of a differential equation, the Noether symmetry is obtained by the requirement that the action integral,

$$W = \int_{t_1}^{t_2} L(q^i, \dot{q}^i, t)dt, \tag{2.1}$$

remains invariant up to a gauge term $f(q^i, \dot{q}^i; t)$. Using the least action principle on (2.1) and simplifying terms upto the first order in ε, gives rise to the condition under which a Noether symmetry,

$$(\xi \frac{d}{dt} + \eta^k \frac{d}{dq^k} + \dot{\eta}^{[k]} \frac{d}{d\dot{q}^k})L + L(q^k, \dot{q}^k, t)\frac{d\xi}{dt} = \frac{df}{dt}, \tag{2.2}$$

exists. In more convenient form Eq. (2.2) can be written as,

$$YL + LD_t\xi = D_t f, \tag{2.3}$$

where Y is the prolonged symmetry generator and D_t the total derivative.
In order to solve the wave equation via its Noether symmetries in a spherically symmetric metric, we choose the metric given by[31]

$$ds^2 = e^{\upsilon(t,r)}dt^2 - e^{\lambda(t,r)}dr^2 - e^{\mu(t,r)}d\theta^2 - e^{\mu(t,r)}\sin^2\theta d\phi^2. \tag{2.4}$$

On this metric the wave equation is written, using the formula, $\Box_g u = \frac{\partial}{\partial x_i}(\sqrt{\mid g \mid} g^{ik}\frac{\partial u}{\partial x_k}) = 0$, where g^{ik} is the inverse and $\mid g \mid$ is the determinant of the metric (2.4). Using (2.4) and simplifying $\Box_g u = 0$, takes the form,

$$\frac{\partial}{\partial t}(e^{(\mu - \frac{\upsilon}{2} + \frac{\lambda}{2})}\sin\theta\frac{\partial u}{\partial t}) - \frac{\partial}{\partial r}(e^{(\mu + \frac{\upsilon}{2} - \frac{\lambda}{2})}\sin\theta\frac{\partial u}{\partial r}) - \frac{\partial}{\partial \theta}(e^{(\frac{\upsilon}{2} + \frac{\lambda}{2})}\sin\theta\frac{\partial u}{\partial \theta}) - \frac{\partial}{\partial \phi}(\frac{e^{(\frac{\upsilon}{2} + \frac{\lambda}{2})}\frac{\partial u}{\partial \phi}}{\sin\theta}) = 0. \tag{2.5}$$

The above equation yields a Lagrangian of the wave equation,

$$2L = e^{(\mu-\frac{\upsilon}{2}+\frac{\lambda}{2})}\sin\theta(\tfrac{\partial u}{\partial t})^2 - e^{(\mu+\frac{\upsilon}{2}-\frac{\lambda}{2})}\sin\theta\big(\tfrac{\partial u}{\partial r}\big)^2 - e^{(\frac{\upsilon}{2}+\frac{\lambda}{2})}\sin(\theta)\big(\tfrac{\partial u}{\partial \theta}\big)^2 - \frac{e^{(\frac{\upsilon}{2}+\frac{\lambda}{2})}\big(\frac{\partial u}{\partial \phi}\big)^2}{\sin\theta}. \tag{2.6}$$

In order to investigate Noether symmetries for given $\upsilon(t, r)$, $\lambda(t, r)$ and $\mu(t, r)$, we assume that the gauge term is independent of derivatives. Throughout the paper we use a convention in which derivatives of u with respect to t, r, θ and ϕ are respectively represented by u_1, u_2, u_3, and u_4.

In order that $Y = \xi_1\partial_t + \xi_2\partial_r + \xi_3\partial_\theta + \xi_4\partial_\phi + \eta\partial_u$ is a Noether point symmetry generator of Eq. (2.5), it must satisfy Eq. (2.3) written explicitly as,

$$YL + L(\xi_{1,t} + u_1\xi_{1,u} + \xi_{2,r} + u_2\xi_{2,u} + \xi_{3,\theta} + u_3\xi_{3,u} + \xi_{4,\phi} + u_4\xi_{4,u}) = B_{1,t} + u_1B_{1,u} + B_{2,r} + u_2B_{2,u} + B_{3,\theta} + u_3B_{3,u} + B_{4,\phi} + u_4B_{4,u}. \tag{2.7}$$

Substituting expression for L from (2.6), applying the operator Y, re-arranging and separating the resulting equation in terms of derivatives of u, the Eq. (2.7) yields an over determined linear system of equations:

$$\xi_{1,u} = 0, \tag{2.8}$$

$$\xi_{2,u} = 0, \tag{2.9}$$

$$\xi_{3,u} = 0, \tag{2.10}$$

$$\xi_{4,u} = 0, \tag{2.11}$$

$$\mathrm{e}^{\nu}\xi_{1,r} - \mathrm{e}^{\lambda}\xi_{2,t} = 0, \tag{2.12}$$

$$\mathrm{e}^{\nu}\xi_{1,\theta} - \mathrm{e}^{\mu}\xi_{3,t} = 0, \tag{2.13}$$

$$\mathrm{e}^{\nu}\xi_{1,\phi} - \mathrm{e}^{\mu}\sin^2\theta\,\xi_{4,t} = 0, \tag{2.14}$$

$$\mathrm{e}^{\lambda}\xi_{2,\theta} + \mathrm{e}^{\mu}\xi_{3,r} = 0, \tag{2.15}$$

$$\mathrm{e}^{\lambda}\xi_{2,\phi} + \mathrm{e}^{\mu}\sin^2\theta\,\xi_{4,r} = 0, \tag{2.16}$$

$$\xi_{3,\phi} + \sin^2\theta\,\xi_{4,\theta} = 0, \tag{2.17}$$

$$B_{1,u} - e^{(\mu-\frac{\upsilon}{2}+\frac{\lambda}{2})}\sin\theta\,\eta_{1,t} = 0, \tag{2.18}$$

$$B_{2,u} + e^{(\mu+\frac{\upsilon}{2}-\frac{\lambda}{2})}\sin\theta\,\eta_{1,r} = 0, \tag{2.19}$$

$$B_{3,u} + e^{(\frac{\upsilon}{2}+\frac{\lambda}{2})}\eta_{1,\theta}\sin\theta = 0, \tag{2.20}$$

$$B_{4,u} + \frac{e^{(\frac{\upsilon}{2}+\frac{\lambda}{2})}\eta_{1,\phi}}{\sin\theta} = 0, \tag{2.21}$$

$$B_{1,t} + B_{2,r} + B_{3,\theta} + B_{4,\phi} = 0, \tag{2.22}$$

$$((\nu_t+\lambda_t)\xi_1+(\nu_r+\lambda_r)\xi_2+2\,\xi_{1,t}+2\,\xi_{2,r}-2\,\xi_{3,\theta}+2\,\xi_{4,\phi}+4\,\eta_{1,u})\sin\theta+2\,\xi_3\cos\theta=0, \tag{2.23}$$

$$((\nu_t+\lambda_t)\xi_1+(\nu_r+\lambda_r)\xi_2+2\,\xi_{1,t}+2\,\xi_{2,r}+2\,\xi_{3,\theta}-2\,\xi_{4,\phi}+4\,\eta_{1,u})\sin\theta-2\,\xi_3\cos\theta=0, \tag{2.24}$$

$$((\nu_t-\lambda_t+2\,\mu_t)\xi_1+(\nu_r-\lambda_r+2\,\mu_r)\xi_2+2\,\xi_{1,t}-2\,\xi_{2,r}+2\,\xi_{3,\theta}+2\,\xi_{4,\phi}+4\,\eta_{1,u})\sin\theta+2\,\xi_3\cos\theta=0, \tag{2.25}$$

$$((\nu_t-\lambda_t-2\,\mu_t)\xi_1+(\nu_r-\lambda_r-2\,\mu_r)\xi_2+2\,\xi_{1,t}-2\,\xi_{2,r}-2\,\xi_{3,\theta}-2\,\xi_{4,\phi}-4\,\eta_{1,u})\sin\theta-2\,\xi_3\cos\theta=0. \tag{2.26}$$

From the above set of Eqs. (2.23)-(2.26) can be transformed to an equivalent system via an invertible transformation,

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} (2.23) \\ (2.24) \\ (2.25) \\ (2.26) \end{pmatrix}, \tag{2.27}$$

The equivalent new equations are:

$$\begin{aligned} &e_1: 2\,\eta_{1,u}+2\,\xi_{1,t}+\xi_1\nu_t+\xi_2\nu_r=0,\\ &e_2: 2\,\eta_{1,u}+2\,\xi_{2,r}+\xi_1\lambda_t+\xi_2\lambda_r=0,\\ &e_3: 2\,\eta_{1,u}+2\,\xi_{3,\theta}+\xi_1\mu_t+\xi_2\mu_r=0,\\ &e_4: 2\,\eta_{1,u}+2\,\xi_{4,\phi}+\xi_1\mu_t+\xi_2\mu_r+2\xi_3\cot\theta=0. \end{aligned} \tag{2.28}$$

From Eqs. (2.28)-(1), (2.8), and (2.9) we deduce that,

$$\eta_1=\alpha(t,r,\theta,\phi)u+\beta(t,r,\theta,\phi). \tag{2.29}$$

By substituting η_1 in Eqs. (2.28), it reduces to the following system:

$$\begin{aligned} &2\,\alpha+2\,\xi_{1,t}+\xi_1\nu_t+\xi_2\nu_r=0,\\ &2\,\alpha+2\,\xi_{2,r}+\xi_1\lambda_t+\xi_2\lambda_r=0,\\ &2\,\alpha+2\,\xi_{3,\theta}+\xi_1\mu_t+\xi_2\mu_r=0,\\ &2\,\alpha+2\,\xi_{4,\phi}+\xi_1\mu_t+\xi_2\mu_r+2\xi_3\cot\theta=0. \end{aligned} \tag{2.30}$$

Moreover, from Eqs. (2.18)-(2.21) and (2.29) we obtain,

$$B_1=\tfrac{1}{2}\sin\theta e^{\mu-\frac{\nu}{2}+\frac{\lambda}{2}}(\alpha_t u^2+2\,\beta_t u)+f_1(t,r,\theta,\phi), \tag{2.31}$$

$$B_2=-\tfrac{1}{2}\sin\theta e^{\mu+\frac{\nu}{2}-\frac{\lambda}{2}}(\alpha_r u^2+2\,\beta_r u)+f_2(t,r,\theta,\phi), \tag{2.32}$$

$$B_3=-\tfrac{1}{2}\sin\theta e^{\frac{\nu}{2}+\frac{\lambda}{2}}(\alpha_\theta u^2+2\,\beta_\theta u)+f_3(t,r,\theta,\phi), \tag{2.33}$$

$$B_4=-\tfrac{1}{2\sin\theta}e^{\frac{\nu}{2}+\frac{\lambda}{2}}(\alpha_\phi u^2+2\,\beta_\phi u)+f_4(t,r,\theta,\phi). \tag{2.34}$$

Also from (2.22) and (2.31)-(2.34), after comparing the different power of u, we find that α and β are solutions for the wave equation and f_1, f_2, f_3, and f_4 satisfy,

$$f_{1,t}+f_{2,r}+f_{3,\theta}+f_{4,\phi}=0. \tag{2.35}$$

Now, we start by evaluating ξ_1, ξ_2, ξ_3, and ξ_4 in terms of explicit functions of θ and ϕ. Taking the sum of the partial derivative of Eqs. (2.13) and (2.14) with respect to ϕ and θ, respectively, and using

Eq. (2.17) in (2.14) we get,

$$D\xi_1 = 0, \tag{2.36}$$

where the operator D is defined as follows:

$$D = \frac{\partial^2}{\partial\theta\partial\phi} - \cot\theta\frac{\partial}{\partial\phi} \tag{2.37}$$

Similarly, by taking the sum of the partial derivative of Eqs. (2.15) and (2.16) with respect to ϕ and θ, respectively, and using Eq. (2.17), we get

$$D\xi_2 = 0. \tag{2.38}$$

Applying the operator D on (2.30)-(1) and using (2.36) and (2.37), we get,

$$D\alpha = 0. \tag{2.39}$$

Also, applying the operator D on (2.30)-(3) and using (2.36), (2.38) and (2.39), we get,

$$D\xi_{3,\theta} = \xi_{3,\phi\theta\theta} - \cot\theta\xi_{3,\phi\theta} = T\xi_{3,\phi} = 0, \tag{2.40}$$

where the operator T is defined as follows:

$$T = \frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta}. \tag{2.41}$$

Applying the operator T on (2.17) and using (2.40), we get,

$$3\cot\theta\xi_{4,\theta\theta} - 2\xi_{4,\theta} + \xi_{4,\theta\theta\theta} = 0, \tag{2.42}$$

Integrating Eq. (2.42) with respect to theta instantly gives,

$$\xi_4 = h_1(t,r,\phi)\,\mathrm{cosec}\,\theta + h_2(t,r,\phi)\cot\theta + h_3(t,r,\phi). \tag{2.43}$$

By subtracting (2.30)-(3) from (2.30)-(4), then solving it give,

$$\xi_3 = -h_{1,\phi}\cos\theta - h_{3,\phi}\sin\theta\tanh^{-1}(\cos\theta) - h_{2,\phi} + h_4(t,r,\phi)\sin\theta \tag{2.44}$$

Substituting above values for ξ_3 and ξ_4 in Eq. (2.17) gives

$$\begin{aligned} h_1 &= k_1(t,r)\sin\phi + k_2(t,r)\cos\phi,\\ h_2 &= k_3(t,r)\sin\phi + k_4(t,r)\cos\phi,\\ h_3 &= k_5(t,r)\phi + k_6(t,r),\\ h_4 &= k_7(t,r). \end{aligned} \tag{2.45}$$

Now, using Eqs. (2.13) and (2.15) give us ξ_1 and ξ_2 in terms of explicit functions of θ

$$\begin{aligned} \xi_1 = {}& e^{\mu-\nu}[\sin\theta(k_{2,t}\sin\phi - k_{1,t}\cos\phi) - k_{7,t}\cos\theta +\\ & k_{5,t}\cos\theta\tanh^{-1}(\cos\theta) + k_{5,t}\ln(\sin\theta) + \theta(k_{4,t}\sin\phi - k_{3,t}\cos\phi)] + h_5(t,r,\phi)\\ \xi_2 = {}& -e^{\mu-\lambda}[\sin\theta(k_{2,r}\sin\phi - k_{1,r}\cos\phi) - k_{7,r}\cos\theta +\\ & k_{5,r}\cos\theta\tanh^{-1}(\cos\theta) + k_{5,r}\ln(\sin\theta) + \theta(k_{4,r}\sin\phi - k_{3,r}\cos\phi)] + h_6(t,r,\phi). \end{aligned} \tag{2.46}$$

Now, substituting the above values of ξ_1 and ξ_4 in Eq. (2.14) yields
$h_5(t,r,\phi) = k_8(t,r),\ k_3(t,r) = w_3(r),\ k_4(t,r) = w_4(r),\ k_5(t,r) = w_5(r),\ k_6(t,r) = w_6(r).$
Using values of ξ_2 and ξ_4 in Eq. (2.16) yields
$h_6(t,r,\phi) = k_9(t,r),\ w_3(r) = c_3,\ w_4(r) = c_4,\ w_5(r) = c_5,\ w_6(r) = c_6.$
Substituting above values of ξ_1 and ξ_2 in Eq. (2.12) yields the following conditions:

$$\begin{aligned} &2k_{1,rt} + (\mu_t - \lambda_t)k_{1,r} + (\mu_r - \nu_r)k_{1,t} = 0,\\ &2k_{2,rt} + (\mu_t - \lambda_t)k_{2,r} + (\mu_r - \nu_r)k_{2,t} = 0,\\ &2k_{7,rt} + (\mu_t - \lambda_t)k_{7,r} + (\mu_r - \nu_r)k_{7,t} = 0,\\ &e^{\lambda}k_{9,t} - e^{\nu}k_{8,r} = 0. \end{aligned} \tag{2.47}$$

Now, Eq. (2.30)-(1) give α in terms of explicit functions of θ and ϕ as follows:

$$\alpha = \tfrac{1}{2}[\sin\theta\cos\phi\ [e^{\mu-\nu}(2\,k_{1,t,t} - k_{1,t}\nu_t + 2\,\mu_t k_{1,t}) - e^{\mu-\lambda}\nu_r k_{1,r}] \\ - \sin\theta\sin\phi\ [e^{\mu-\nu}(2\,k_{2,t,t} - \nu_t k_{2,t} + 2\,\mu_t k_{2,t}) - e^{\mu-\lambda}\nu_r k_{2,r}] \\ + \cos\theta\ [e^{\mu-\nu}(2\,k_{7,t,t} - k_{7,t}\nu_t + 2\,\mu_t k_{7,t}) - e^{\mu-\lambda}\nu_r k_{7,r}]] - \tfrac{1}{2}(\nu_t k_8 + \nu_r k_9 + 2\,k_{8,t}). \tag{2.48}$$

Equation (2.30)-(2) give us the following conditions:

$$\begin{aligned} &2\,e^{\nu}k_{1,rr} + 2\,e^{\lambda}k_{1,tt} + e^{\nu}(2\mu_r - \nu_r - \lambda_r)k_{1,r} + e^{\lambda}(2\mu_t - \nu_t - \lambda_t)\,k_{1,t} = 0,\\ &2\,e^{\nu}k_{2,rr} + 2\,e^{\lambda}k_{2,tt} + e^{\nu}(2\mu_r - \nu_r - \lambda_r)k_{2,r} + e^{\lambda}(2\mu_t - \nu_t - \lambda_t)\,k_{2,t} = 0,\\ &2\,e^{\nu}k_{7,rr} + 2\,e^{\lambda}k_{7,tt} + e^{\nu}(2\mu_r - \nu_r - \lambda_r)k_{7,r} + e^{\lambda}(2\mu_t - \nu_t - \lambda_t)\,k_{7,t} = 0,\\ &\nu_r k_9 - \lambda_t k_8 - \lambda_r k_9 - 2\,k_{9,r} + \nu_t k_8 + 2\,k_{8,t} = 0. \end{aligned} \tag{2.49}$$

Equation (2.30)-(3) give us that $c_5 = 0$ with the following conditions:

$$\begin{aligned} &2\,e^{\mu}e^{\lambda}k_{1,tt} + e^{\mu}e^{\nu}(\mu_r - \nu_r)k_{1,r} + e^{\mu}e^{\lambda}(\mu_t - \nu_t)k_{1,t} + 2e^{\lambda}e^{\nu}k_1 = 0,\\ &2\,e^{\mu}e^{\lambda}k_{2,tt} + e^{\mu}e^{\nu}(\mu_r - \nu_r)k_{2,r} + e^{\mu}e^{\lambda}(\mu_t - \nu_t)k_{2,t} + 2e^{\lambda}e^{\nu}k_2 = 0,\\ &2\,e^{\mu}e^{\lambda}k_{7,tt} + e^{\mu}e^{\nu}(\mu_r - \nu_r)k_{7,r} + e^{\mu}e^{\lambda}(\mu_t - \nu_t)k_{7,t} + 2e^{\lambda}e^{\nu}k_7 = 0,\\ &\mu_t k_8 + \mu_r k_9 - \nu_t k_8 - \nu_r k_9 - 2\,k_{8,t} = 0. \end{aligned} \tag{2.50}$$

Finally, since α defined in Eq. (2.48) is a solution for the wave Eq. (2.5), the final solution, upto known functions of θ and ϕ, becomes:

$$\begin{aligned} \xi_1 &= e^{\mu-\nu}[\sin\theta(k_{2,t}\sin\phi - k_{1,t}\cos\phi) - k_{7,t}\cos\theta] + k_8,\\ \xi_2 &= -e^{\mu-\lambda}[\sin\theta(k_{2,r}\sin\phi - k_{1,r}\cos\phi) - k_{7,r}\cos\theta] + k_9,\\ \xi_3 &= (k_2\sin\phi - k_1\cos\phi)\cos\theta + k_7\sin\theta + c_4\sin\phi - c_3\cos\phi,\\ \xi_4 &= (k_1\sin\phi + k_2\cos\phi)\csc\theta + (c_3\sin\phi + c_4\cos\phi)\cot\theta + c_6. \end{aligned} \tag{2.51}$$

subject to some conditions on $k_1, k_2, k_7, k_8,$ and k_9. At this stage the problem is reduced to finding eight functions of two variables (t and r), namely, $k_1, k_2, k_7, k_8, k_9, \nu, \lambda$, and μ given in Eqs. (2.47), (2.49), (2.50) and Appendix A. Note that ν and λ in the above system are arbitrary functions of temporal and radial coordinates. It is, therefore, not easy to solve the unknowns explicitly in terms of these variables for arbitrary metrics.

III. THE WAVE EQUATION ON THE FRIEDMANN ROBERTSON WALKER UNIVERSE

The Friedmann Robertson Walker universe is described by the line element[31]

$$ds^2 = dt^2 - a(t)^2(\tfrac{dr^2}{1-kr^2} + r^2 d\Omega^2), \tag{3.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, $a(t)$ is the scale factor and k is the curvature parameter with $k = -1, 0, 1$ corresponding to open, flat, and closed universes, respectively. Thus the wave Eq. (2.5) on this metric (with $\nu = 0$, $\lambda = 2\ln\left(\frac{a(t)}{\sqrt{1-kr^2}}\right)$, $\mu = \ln(r^2 a(t)^2)$) takes the following form:

$$\tfrac{\partial}{\partial t}(\tfrac{r^2 a(t)^3\sin\theta}{\sqrt{1-kr^2}}\tfrac{\partial u}{\partial t}) - \tfrac{\partial}{\partial r}(r^2 a(t)\sin\theta\sqrt{1-kr^2}\tfrac{\partial u}{\partial r}) - \tfrac{\partial}{\partial\theta}(\tfrac{a(t)\sin\theta}{\sqrt{1-kr^2}}\tfrac{\partial u}{\partial\theta}) - \tfrac{\partial}{\partial\phi}(\tfrac{a(t)}{\sin\theta\sqrt{1-kr^2}}\tfrac{\partial u}{\partial\phi}) = 0. \tag{3.2}$$

A. Flat universe

Since the value of the scale factor for the flat universe, when $k = 0$, is given as $a(t) = t^{\frac{2}{3}}$, The wave equation (3.2) takes the form,

$$\tfrac{\partial}{\partial t}(r^2 t^2\sin\theta\tfrac{\partial u}{\partial t}) - \tfrac{\partial}{\partial r}(r^2 t^{\frac{2}{3}}\sin\theta\tfrac{\partial u}{\partial r}) - \tfrac{\partial}{\partial\theta}(t^{\frac{2}{3}}\sin\theta\tfrac{\partial u}{\partial\theta}) - \tfrac{\partial}{\partial\phi}(\tfrac{t^{\frac{2}{3}}}{\sin\theta}\tfrac{\partial u}{\partial\phi}) = 0. \tag{3.3}$$

Now by solving the system given by Eqs. (2.47)–(2.50), we will find the Noether symmetries for the above wave equation as follow:

From Eq. (2.47)(1) we get,

$$k_1 = F_1(r) + \frac{F_2(t)}{r}. \tag{3.4}$$

From Eqs. (2.49)(1) and (2.50)(1) we get,

$$r^2 k_{1,rr} + r k_{1,r} - k_1 = 0. \tag{3.5}$$

Substituting Eq. (3.4) in Eq. (3.5) gives the following Euler equation:

$$r^2 F_{1,rr} + r F_{1,r} - F_1 = 0. \tag{3.6}$$

Thus F_1 is given as follows:

$$F_1(r) = d_1 r + \frac{d_2}{r}. \tag{3.7}$$

Substituting Eqs. (3.4) and (3.7) in Eq. (2.50)(1) gives,

$$t^{\frac{4}{3}} F_{2,tt} + \frac{2}{3} t^{\frac{1}{3}} F_{2,t} + 2\, d_1 = 0. \tag{3.8}$$

Correspondingly the solution F_2 is given as follows:

$$F_2(t) = 3 d_3 t^{\frac{1}{3}} - 9 d_1 t^{\frac{2}{3}} + C. \tag{3.9}$$

Substituting Eqs. (3.7) and (3.9) in Eq. (3.4) gives us

$$k_1 = d_1 (r - \frac{9}{r} t^{\frac{2}{3}}) + \frac{d_2}{r} + 3 \frac{d_3}{r} t^{\frac{1}{3}}. \tag{3.10}$$

Similarly, we get that

$$\begin{aligned} k_2 &= d_4 (r - \tfrac{9}{r} t^{\frac{2}{3}}) + \tfrac{d_5}{r} + 3 \tfrac{d_6}{r} t^{\frac{1}{3}}, \\ k_7 &= d_7 (r - \tfrac{9}{r} t^{\frac{2}{3}}) + \tfrac{d_8}{r} + 3 \tfrac{d_9}{r} t^{\frac{1}{3}}. \end{aligned} \tag{3.11}$$

Also from Eqs. (2.49)(4) and (2.50)(4) we get

$$k_9 - r k_{9,r} = 0, \tag{3.12}$$

giving k_9 as follows:

$$k_9 = W_1(t) r. \tag{3.13}$$

Substituting Eq. (3.13) in Eq. (2.47)(4) gives us

$$k_8 = \frac{r^2}{2} t^{\frac{4}{3}} W_{1,t} + W_2(t). \tag{3.14}$$

Substituting Eqs. (3.13) and (3.14) in Eq. (2.50)(4) gives the following two equations:

$$\begin{aligned} & t^{\frac{2}{3}} W_{1,tt} + \tfrac{2}{3} t^{-\frac{1}{3}} W_{1,t} = 0, \\ & t^{\frac{2}{3}} W_{2,t} - \tfrac{2}{3} t^{-\frac{1}{3}} W_2(t) - t^{\frac{2}{3}} W_1(t) = 0. \end{aligned} \tag{3.15}$$

So W_1 and W_2 are given as follows:

$$\begin{aligned} W_1 &= d_{10} + 6 d_{11} t^{\frac{1}{3}}, \\ W_2 &= 3 d_{10} t + 9 d_{11} t^{\frac{4}{3}} + d_{12} t^{\frac{2}{3}}. \end{aligned} \tag{3.16}$$

Substituting Eq. (3.16) in Eqs. (3.13) and (3.14) gives,

$$\begin{aligned} k_8 &= 3 d_{10} t + d_{11} (r^2 t^{\frac{2}{3}} + 9 t^{\frac{4}{3}}) + d_{12} t^{\frac{2}{3}}, \\ k_9 &= d_{10} r + 6 d_{11} r t^{\frac{1}{3}}. \end{aligned} \tag{3.17}$$

Finally, the solution of the system (2.47)–(2.50) for this metric is summarized as follows:

$$\begin{aligned} k_1 &= d_1(r - \tfrac{9}{r}t^{\frac{2}{3}}) + \tfrac{d_2}{r} + 3\tfrac{d_3}{r}t^{\frac{1}{3}}, \\ k_2 &= d_4(r - \tfrac{9}{r}t^{\frac{2}{3}}) + \tfrac{d_5}{r} + 3\tfrac{d_6}{r}t^{\frac{1}{3}}, \\ k_7 &= d_7(r - \tfrac{9}{r}t^{\frac{2}{3}}) + \tfrac{d_8}{r} + 3\tfrac{d_9}{r}t^{\frac{1}{3}}, \\ k_8 &= 3d_{10}t + d_{11}(r^2t^{\frac{2}{3}} + 9t^{\frac{4}{3}}) + d_{12}t^{\frac{2}{3}}, \\ k_9 &= d_{10}r + 6d_{11}rt^{\frac{1}{3}}. \end{aligned} \tag{3.18}$$

Now using constraints in Appendix A, we get the following components of the Noether symmetries:

$$\begin{aligned} Y_1 &= 6\,rt\sin\theta\cos\phi\tfrac{\partial}{\partial t} + (9t^{\frac{2}{3}} + r^2)\sin\theta\cos\phi\tfrac{\partial}{\partial r} + \tfrac{(9t^{\frac{2}{3}}-r^2)}{r}\cos\theta\cos\phi\tfrac{\partial}{\partial\theta} - \tfrac{(9t^{\frac{2}{3}}-r^2)}{r\sin\theta}\sin\phi\tfrac{\partial}{\partial\phi} \\ &\quad -6\,ru\sin\theta\cos\phi\tfrac{\partial}{\partial u}, \\ Y_2 &= 6\,rt\sin\theta\sin\phi\tfrac{\partial}{\partial t} + (9t^{\frac{2}{3}} + r^2)\sin\phi\sin\theta\tfrac{\partial}{\partial r} + \tfrac{(9t^{\frac{2}{3}}-r^2)}{r}\cos\theta\sin\phi\tfrac{\partial}{\partial\theta} + \tfrac{(9t^{\frac{2}{3}}-r^2)}{r\sin\theta}\cos\phi\tfrac{\partial}{\partial\phi} \\ &\quad -6\,ru\sin\theta\sin\phi\tfrac{\partial}{\partial u}, \\ Y_3 &= 6\,rt\cos\theta\tfrac{\partial}{\partial t} + (9t^{\frac{2}{3}} + r^2)\cos\theta\tfrac{\partial}{\partial r} - \tfrac{(9t^{\frac{2}{3}}-r^2)}{r}\sin\theta\tfrac{\partial}{\partial\theta} - 6\,ru\cos\theta\tfrac{\partial}{\partial u}, \\ Y_4 &= 3t\tfrac{\partial}{\partial t} + r\tfrac{\partial}{\partial r} - 3u\tfrac{\partial}{\partial u}, \\ Y_5 &= -\sin\theta\cos\phi\tfrac{\partial}{\partial r} - \tfrac{\cos\theta\cos\phi}{r}\tfrac{\partial}{\partial\theta} + \tfrac{\sin\phi}{r\sin\theta}\tfrac{\partial}{\partial\phi}, \\ Y_6 &= \sin\phi\sin\theta\tfrac{\partial}{\partial r} + \tfrac{\cos\theta\sin\phi}{r}\tfrac{\partial}{\partial\theta} + \tfrac{\cos\phi}{r\sin\theta}\tfrac{\partial}{\partial\phi}, \\ Y_7 &= -\cos\theta\tfrac{\partial}{\partial r} + \tfrac{\sin\theta}{r}\tfrac{\partial}{\partial\theta}, \\ Y_8 &= -\cos\phi\tfrac{\partial}{\partial\theta} + \tfrac{\cos\theta}{\sin\theta}\sin\phi\tfrac{\partial}{\partial\phi}, \\ Y_9 &= \sin\phi\tfrac{\partial}{\partial\theta} + \tfrac{\cos\theta}{\sin\theta}\cos\phi\tfrac{\partial}{\partial\phi}, \\ Y_{10} &= \tfrac{\partial}{\partial\phi}, \\ Y_\beta &= \beta\tfrac{\partial}{\partial u}. \end{aligned} \tag{3.19}$$

The commutation relations of the Lie algebra of the ten Noether symmetries are given in Table I of Appendix B.

B. Linearization of a (3 + 1) wave equation on the flat universe

In this section we transform the wave equation (3.3) on the Friedmann flat metric, from linear PDE with variable coefficients to linear PDE with constant coefficients with respect to the derivative of the dependent variable by using a specific subalgebra of the above Noether symmetries.
Defining invertible transformations

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, $\tau = -3t^{\frac{1}{3}}$ and $w = t^{\frac{2}{3}}u$, transforms the subalgebra Y_4, Y_5, Y_6 and Y_7 to:

$$\begin{aligned} Y_4 &= \tau\tfrac{\partial}{\partial\tau} + x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y} + z\tfrac{\partial}{\partial z} - w\tfrac{\partial}{\partial w}, \\ Y_5 &= -\tfrac{\partial}{\partial x}, \\ Y_6 &= \tfrac{\partial}{\partial y}, \\ Y_7 &= -\tfrac{\partial}{\partial z}. \end{aligned} \tag{3.20}$$

The other symmetries transform (under these transformations) to:

$$\begin{aligned} Y_1 &= 2x\tau\tfrac{\partial}{\partial\tau} + (x^2 - y^2 - z^2 + \tau^2)\tfrac{\partial}{\partial x} + 2xy\tfrac{\partial}{\partial y} + 2xz\tfrac{\partial}{\partial z} - 2xw\tfrac{\partial}{\partial w}, \\ Y_2 &= 2y\tau\tfrac{\partial}{\partial\tau} + 2yx\tfrac{\partial}{\partial x} + (y^2 - x^2 - z^2 + \tau^2)\tfrac{\partial}{\partial y} + 2yz\tfrac{\partial}{\partial z} - 2yw\tfrac{\partial}{\partial w} \\ Y_3 &= 2z\tau\tfrac{\partial}{\partial t} + 2zx\tfrac{\partial}{\partial x} + 2zy\tfrac{\partial}{\partial y} + (z^2 - x^2 - y^2 + \tau^2)\tfrac{\partial}{\partial z} - 2zw\tfrac{\partial}{\partial w} \\ Y_8 &= -z\tfrac{\partial}{\partial x} + x\tfrac{\partial}{\partial z}, \\ Y_9 &= z\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial z}, \\ Y_{10} &= -y\tfrac{\partial}{\partial x} + x\tfrac{\partial}{\partial y}, \\ X_\beta &= \beta\tfrac{\partial}{\partial u}. \end{aligned} \tag{3.21}$$

Under the coordinate transformation $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$, the wave equation (3.3) takes the form,

$$\frac{\partial}{\partial t}(t^2 u_t) = t^{\frac{2}{3}}\nabla^2 u. \tag{3.22}$$

If we use the transformation $w = t^{\frac{2}{3}}u$, the wave equation (3.22) becomes,

$$\frac{\partial}{\partial t}(t^{\frac{2}{3}} w_t) = t^{-\frac{2}{3}}(\nabla^2 w + \tfrac{2}{9}t^{-\frac{2}{3}}). \tag{3.23}$$

Then by using the new variable $\tau = -3t^{\frac{1}{3}}$, the equation is transformed again to,

$$w_{\tau\tau} = \nabla^2 w + \frac{2}{\tau^2}w. \tag{3.24}$$

Now, using the method of separation of variable by taking $w = f(\tau)v(x, y, z)$, the above equation gives an ordinary differential equation of second order,

$$\tau^2\frac{d^2}{d\tau^2}f(\tau) + (\lambda^2\tau^2 - 2)f(\tau) = 0, \tag{3.25}$$

whose general solution is,

$$f(\tau) = \frac{C_1\,(\lambda\,\tau\cos(\lambda\,\tau) - \sin(\lambda\,\tau))}{\tau} + \frac{C_2\,(\cos(\lambda\,\tau) + \lambda\,\tau\sin(\lambda\,\tau))}{\tau}, \tag{3.26}$$

and the Helmholtz equation,

$$\nabla^2 v + \lambda^2 v = 0. \tag{3.27}$$

The solution of Eq. (3.24) can be constructed for some specific initial and boundary conditions.

IV. CONCLUSION

In this paper we found the Noether symmetries of a $(3 + 1)$ wave equation on the general spherical metric explicitly in terms of the explicit functions of θ and ϕ. In order to solve the Noether symmetries in terms of known functions of all the spacetime variables we chose a specific flat Friedmann metric. We get ten Noether symmetries which are three translations, three rotations, one-parameter dilation group, Y_1, Y_2, and Y_3. Finally, we have converted the wave equation with variable coefficients to the one with constant coefficients with respect to the derivative of the dependent variable by using invertible transformations. The transformed wave equation is finally converted to a second order differential equation and Helmoltz equation.

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APPENDIX A: ADDITIONAL DIFFERENTIAL CONSTRAINTS SATISFIED BY NOETHER SYMMETRIES

$$2e^{\mu+\nu}R_{1,rr} - 2\,e^{\mu+\lambda}R_{1,tt} + e^{\mu+\nu}(2\mu_r + \nu_r - \lambda_r)R_{1,r} - e^{\mu+\lambda}(2\mu_t - \nu_t + \lambda_t)R_{1,t} - 4e^{\nu+\lambda}R_1 = 0,$$
$$2\,e^{\mu+\nu}R_{2,rr} - 2\,e^{\mu+\lambda}R_{2,tt} + e^{\mu+\nu}(2\mu_r + \nu_r - \lambda_r)R_{2,r} - e^{\mu+\lambda}(2\mu_t - \nu_t + \lambda_t)R_{2,t} - 4e^{\nu+\lambda}R_2 = 0,$$
$$2e^{\mu+\nu}R_{3,rr} - 2e^{\mu+\lambda}R_{3,tt} + e^{\mu+\nu}(2\mu_r + \nu_r - \lambda_r)R_{3,r} - e^{\mu+\lambda}(2\mu_t - \nu_t + \lambda_t)R_{3,t} - 4e^{\nu+\lambda}R_3 = 0,$$
$$2e^{\nu}R_{4,rr} - 2\,e^{\lambda}R_{4,tt} + e^{\nu}(2\mu_r + \nu_r - \lambda_r)R_{4,r} - e^{\lambda}(2\mu_t - \nu_t + \lambda_t)R_{4,t} = 0,$$

where

$$R_1(t,r) = e^{\mu-\nu}(2\,k_{1,t,t} - k_{1,t}\nu_t + 2\,\mu_t k_{1,t}) - e^{\mu-\lambda}\nu_r k_{1,r},$$
$$R_2(t,r) = e^{\mu-\nu}(2\,k_{2,t,t} - \nu_t k_{2,t} + 2\,\mu_t k_{2,t}) - e^{\mu-\lambda}\nu_r k_{2,r},$$
$$R_3(t,r) = e^{\mu-\nu}(2\,k_{7,t,t} - k_{7,t}\nu_t + 2\,\mu_t k_{7,t}) - e^{\mu-\lambda}\nu_r k_{7,r},$$
$$R_4(t,r) = \nu_t k_8 + \nu_r k_9 + 2\,k_{8,t}.$$

APPENDIX B: COMMUTATION RELATIONS SATISFIED BY THE NOETHER SYMMETRIES OF THE LAGRANGIAN OF THE WAVE EQ. (3.3)

TABLE I. Commutator table for the Lie algebra.

$[Y_i, Y_j]$	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}
Y_1	0	0	0	$-Y_1$	$2Y_4$	$-2Y_{10}$	$2Y_8$	Y_3	0	Y_2
Y_2	0	0	0	$-Y_2$	0	$-2Y_4$	$-2Y_9$	0	$-Y_3$	$-Y_1$
Y_3	0	0	0	$-Y_3$	$-2Y_8$	$-2Y_9$	$2Y_4$	$-Y_1$	Y_2	0
Y_4	Y_1	Y_2	Y_3	0	$-Y_5$	$-Y_6$	$-Y_7$	0	0	0
Y_5	$-2Y_4$	0	$2Y_8$	Y_5	0	0	0	Y_7	0	$-Y_6$
Y_6	$2Y_{10}$	$2Y_4$	$2Y_9$	Y_6	0	0	0	0	Y_7	Y_5
Y_7	$-2Y_8$	$2Y_9$	$-2Y_4$	Y_7	0	0	0	$-Y_5$	$-Y_6$	0
Y_8	$-Y_3$	0	Y_1	0	$-Y_7$	0	Y_5	0	Y_{10}	$-Y_9$
Y_9	0	Y_3	$-Y_2$	0	0	$-Y_7$	Y_6	$-Y_{10}$	0	Y_8
Y_{10}	$-Y_2$	Y_1	0	0	Y_6	$-Y_5$	0	Y_9	$-Y_8$	0

[1] A. Aijaz, A. H. Bokhari, A. H. Kara, and F. D. Zaman, J. Math. Anal. Appl. **339**, 175 (2008).
[2] A. D. Polyanin, *Handbook of Linear Partial Differential Equations for Engineers and Scientists* (CRC, Boca Raton, 2002).
[3] A. G. Johnpillai, A. H. Kara, and F. M. Mahomed, “Conservation laws of a nonlinear (1 + 1) wave equation,” Nonlinear Anal. B. **11**, 2237 (2010).
[4] A. H. Bokhari and A. H. Kara, Gen. Relat. Gravit. **39**, 2053 (2007).
[5] A. H. Bokhari, A. H. Kara, M. Karim, and F. D. Zaman, Int. J. Theor. Phys. **48**(7), 1919 (2009).
[6] A. H. Kara and F. M. Mahomed, J. Nonlinear Math. Phys. **9**, 60 (2002).
[7] A. H. Kara and F. M. Mahomed, Nonlinear Dyn. **45**, 367 (2006).
[8] A. H. Kara and F. M. Mahomed, Int. J. Theor. Phys. **39**, 23 (2000).
[9] A. H. Kara and F. M. Mahomed, “Action of Lie-Backlund symmetries on conservation laws,” *Proceedings of Modern Group Analysis MORGAN VII*, Eds: N. H. Ibragimov, Naqvi and E. R. Straume, (Trondheim Press, Norway, 1997) 175.
[10] A. Sjöberg, Appl. Math. Comput. **184**, 608 (2007).
[11] A. Sjöberg, Nonlinear Anal.: Real World Appl. **10**, 3472 (2009).
[12] B. J. Cantwell, *An Introduction to Symmetry Analysis* (Cambridge University Press, Cambridge, 2002).
[13] C. M. Khalique and F. M. Mahomed, Math. Probl. Eng. **26**(1), 141 (2005).
[14] E. Bessel-Hagen, Math. Ann. **84**, 258 (1921).
[15] E. Noether, Göttingen, Math. Phys. Kl. Heft **2**, 235 (1918) [Transp. Theory Stat. Phys. **1**(3), 186 (1971) (in English)].
[16] G. Bluman and S. C. Anco, *Symmetry and Integration Methods for Differential Equations* (Springer, New York, 2002).
[17] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations* (Springer, New York, 1989).
[18] L. V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic, New York, 1982).
[19] M. I. Servo, Nonlinear Math. Phys. **3**(1-2), 63 (1996).
[20] “Symmetries, exact solutions and conservation laws,” in *CRC Handbook of Lie Group Analysis of Differential Equations*, edited by N. H. Ibragimov (CRC, Boca Raton, 1994), Vol. 1.
[21] “Applications in engineering and physical sciences,” in *CRC Handbook of Lie Group Analysis of Differential Equations*, edited by N. H. Ibragimov (CRC, Boca Raton, 1995), Vol. 2.
[22] “New trends in theoretical developments and computational methods,” in *CRC Handbook of Lie Group Analysis of Differential Equations*, edited by N. H. Ibragimov (CRC, Boca Raton, 1996), Vol. 3.
[23] N. H. Ibragimov and T. Kolsrud, Nonlinear Dyn. **36**(1), 29 (2004).
[24] N. H. Ibragimov, A. H. Kara, and F. M. Mahomed, Nonlinear Dyn. **15**, 115 (1998).
[25] N. H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations*, (Wiley, New York, 1999).
[26] N. H. Ibragimov, *Selected Works* (ALGA, Karlskrona, Sweden 2008), Vol. 3.
[27] P. A. Clarkson and E. L. Mansfield, Physica **70**, 250 (1993).
[28] P. G. Estevez and S. L. Zhang, J. Math. Anal. Appl. **275**, 44 (2002).

[29] P. W. Doyle and P. J. Vassiliou, Int. J. Non-Linear Mech. **33**(2), 315 (2002).
[30] S. C. Anco and G. W. Bluman, J. Math. Anal. Appl. **322**, 233 (2006).
[31] S. M. Carroll, *Spacetime and Geometry* (Addison-Wesley, New York, 2004).
[32] S. C. Anco and G. W. Bluman, Eur. J. Appl. Math. **9**, 567 (2002).
[33] W. F. Ames, R. J. Lohner, and E. Adams, Int. J. Non-Linear Mech. **16**, 439 (1981).
[34] A.Y. Al-Dweik, Invariant solutions, double reductions and conservation laws for certain Evolution equations, Ph.D. Dissertation KFUPM (2010).