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Abstract

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N coupled nonlinear Schrödinger equations: Special set and applications to $N=3$

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Analytic solutions for a special set of N coupled nonlinear Schrödinger equations characterized by 2^N interaction types are presented for $N=1-3$. They provide an overview of the important role played by nonlinear couplings. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1521238]

I. INTRODUCTION

There are many physical processes that are described by the nonlinear Schrödinger equations. Because of problems of mathematical interest and a growing interest in nonlinear phenomena in optical fibers,[1] systems of coupled nonlinear Schrödinger (CNLS) equations involving two and generally N components have attracted a great deal of attention. The complex wavefunction $\phi_m(z,t)$ of the mth component as a function of position z and time t is assumed to satisfy the following N CNLS equations:

$$i\phi_{mz}+\varepsilon_m\phi_{mtt}+\kappa_m\phi_m+\left(\sum_{j=1}^{N}\lambda_{mj}|\phi_j|^2\right)\phi_m=0,\quad m=1,\ldots,N, \tag{1}$$

where ε_m, κ_m and λ_{mj} are real parameters characteristic of the medium and interaction, and where the subscripts in z and t denote derivatives with respect to z and t, and the subscript m is for different components. Although the term $\kappa_m\phi_m$ can be eliminated by a substitution $\phi_m \to \phi_m e^{i\kappa_m z}$, we shall keep the term for the purpose of ordering the N equations that will become clear later.

Soliton[1–4] and solitary-wave solutions[4–6] have been presented for special sets of parameters ε_m, κ_m and λ_{mj}. In particular, Lakshmanan and his collaborators[7] have studied a special set of "mixed" interaction cases using the Painlevé analysis.

In a recent paper, Hioe and Salter[8] showed that there is a special set that can be characterized by 2^N specific arrays of interaction parameters for which the N CNLS equations possess special analytic solutions. For this special set, the N components of the CNLS equations can be expressed in terms of N Lamé functions,[9] and every Lamé function of order $n\leqslant N$ is a solution for one or more components for one or more interaction types of this special set. In particular, they presented simple rules that (a) identify a given combination of Lamé functions to be a solution to one or more specific interaction types, and (b) give all the possible combinations of Lamé functions as solutions of a specific interaction type. These CNLS equations that have these Lamé functions as solutions were called the L-set. It was pointed out that the L-set coincides with the set of CNLS equations that pass the Painlevé test identified by Radhakrishnan *et al.*[7] In a separate development, Hioe and Carroll[10] showed that N coupled Gross–Pitaevskii (CGP) equations in D-dimensions with a square-well and Coulomb potential can be transformed into N CNLS equations, and solutions in terms of same or different Lamé functions can be used to suggest possible ways of making multiple Bose–Einstein condensates overlap each other or separate.

In this article, we first review briefly the L-set and the subsets of it that consist of what are called the weakly and strongly mixed interaction types. We then present and consider analytic

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solutions particularly for $N=3$. The use of two or three coupled waves can play a very important role in optical communication with unexpected results. For example, a wave that cannot be solitary by itself when it propagates through an anomalous group-velocity dispersion (GVD) region[1] can be made to do so when it is coupled to a second appropriate wave, and two waves that cannot be solitary by themselves when they propagate through an anomalous GVD region can be made to do so when they are coupled to a third appropriate wave. For the many analytic solutions presented in this article, besides their theoretical and mathematical interest, the true test of their practical usefulness could be realized if and when a general recognizable pattern would emerge that would strongly suggest the robustness of certain combinations of waveforms.

II. THE L-SET FOR *N* CNLS EQUATIONS

We first review the L-set for a general value of N. The L-set of N CNLS equations (1) is defined to be a special subset of Eq. (1) given by the following,[8]

$$i\phi_{mz}\pm\beta_m\phi_{mtt}+\kappa_m\phi_m\pm\left(\sum_{j=1}^{N}\beta_m\beta_j|\phi_j|^2\right)\phi_m=0,\quad m=1,\dots,N, \tag{2}$$

where $\beta_j=+1$ or -1, for $j=1,\dots,N$. That is, the L-set of N CNLS equations (1) has those coefficients ε_m and λ_{mj} which are expressible as in Eq. (2).

Consider the stationary-wave solution of the form

$$\phi_m(z,t)=\psi_m(t)\exp(i\omega z), \tag{3}$$

where ω is a real constant and $\psi_m(t)$'s are real functions of t only. Substitution of Eq. (3) into Eq. (2) gives the following N equations for $\psi_m(t)$:

$$\psi_{mtt}+c_m\psi_m+\left(\sum_{j=1}^{N}\beta_j\psi_j^2\right)\psi_m=0,\quad m=1,\dots,N, \tag{4}$$

where

$$c_m=\pm\beta_m(\kappa_m-\omega). \tag{5}$$

To eliminate the permutation symmetry, Eq. (4) is arranged such that

$$c_1\geqslant c_2\geqslant\cdots\geqslant c_N, \tag{6}$$

so that only one of the two choices (the upper or lower sign) in Eq. (2) gives Eq. (4). The traveling waves, if required, can be constructed by substituting the solutions ψ_m from Eq. (4) into Eq. (2), and replacing $\phi_m(z,t)$ by $\phi_m(z,t-z/v)\exp\{i[t-z/(2v)]/(2v)\}$, where v is the common velocity of the waves.

The interaction parameters of Eq. (2) [and Eq. (4)] are now characterized by the array $(\beta_1,\beta_2,\dots,\beta_N)$, where $\beta_j=+1$ or -1, and each of the 2^N arrays is referred to as an interaction type. The two special cases are the normal GVD region characterized by $(--\dots-)$ and the anomalous GVD region characterized by $(++\dots+)$. Note that the 2^N interaction types constitute only a subset of the symmetric λ_{mn} that can take on values $+1$ or -1, because the λ_{mn}'s are constrained to be only those given by $\lambda_{mn}=+\beta_m\beta_n$ or $-\beta_m\beta_n$. This special set of CNLS equations that consists of the 2^N interaction types is the L-set.

A Lamé equation of order n can be written in the form[9]

$$d^2f/d\tau^2+[h-n(n+1)k^2sn^2(\tau,k)]f=0. \tag{7}$$

Only the polynomial solutions of the Lamé equation are used here and are referred to as Lamé functions, and the $2n+1$ Lamé functions of order n, $f_1^{(n)},f_2^{(n)},\dots,f_{2n+1}^{(n)}$, are numbered in the order of numbering their corresponding eigenvalues $h_m^{(n)}$ arranged in descending order $h_1^{(n)} > h_2^{(n)} > \cdots > h_{2n+1}^{(n)}$. These Lamé functions and their corresponding eigenvalues, which will be needed later for our analytic solutions, are given in Appendix A for $n=1-3$. An N-combination $(f_{p_1}^{(n)}, f_{p_2}^{(n)}, \ldots, f_{p_N}^{(n)})$ that gives an analytic solution for the N components $(\psi_1, \psi_2, \ldots, \psi_N)$ will be represented simply by $(p_1, p_2, \ldots, p_N)_n$, where Eq. (6) implies $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_N$. The $2n+1$ eigenvalues of the Lamé equation $h_1^{(n)}, h_2^{(n)}, \ldots, h_{2n+1}^{(n)}$ are renumbered as $h_1^{(n)}, h_2^{(n)}, h_{2'}^{(n)}, \ldots, h_{n+1}^{(n)}, h_{(n+1)'}^{(n)}$, and the corresponding Lamé functions $f_1^{(n)}$, $f_2^{(n)}$, $\ldots$, $f_{2n+1}^{(n)}$ as $f_1^{(n)}, f_2^{(n)}, f_{2'}^{(n)}, \ldots, f_{n+1}^{(n)}, f_{(n+1)'}^{(n)}$, i.e., they are grouped in pairs except the first one. The use of Lamé function ansatz described in Ref. 4 gives the required specific values for the c_m of Eq. (4) and the amplitude C_m for the mth component, thus giving, for combination $(p_1, p_2, \ldots, p_N)_n$, the solution

$$\psi_m(t) = C_m f_{p_m}^{(n)}(t), \quad m=1,\ldots,N. \tag{8}$$

Two separate cases of representing the solutions in terms of Lamé functions may be distinguished: case (I) is in terms of Lamé functions of order $n=N$, and case (II) in terms of Lamé functions of order $n<N$.

(I) For $n=N$, the N Lamé functions for the N components must necessarily be different Lamé functions. The number of N combinations that can be chosen from $2n+1$ distinct Lamé functions of order $n=N$, with no repetition allowed, is $M=\binom{2n+1}{N}$.

Every one of the M combinations is a solution of one and only one particular interaction type, and every interaction type has one or more combinations as solutions. More specifically,

(a) Combination $(p_1, p_2+1, p_3+2, \ldots, p_N+N-1)_n$ is a solution of interaction type $((-1)^{p_1}, (-1)^{p_2}, (-1)^{p_3}, \ldots, (-1)^{p_N})$, and

(b) an interaction type $((-1)^{p_1}, (-1)^{p_2}, (-1)^{p_3}, \ldots, (-1)^{p_N})$ has solutions given by all possible combinations $(m_1, m_2, m_3, \ldots, m_N)_n$, that can be obtained by setting $m_1=p_1$, $m_2=p_2+1$, $m_3=p_3+2, \ldots$, $m_N=p_N+N-1$, and by increasing or decreasing m_j by any multiple of 2 subject to the condition that $m_1<m_2<m_3<\cdots<m_N$, where m_j can take on values $1,2,2',3,3',\ldots,n+1,(n+1)'$ with the understanding that $r<r'$.

(II) For $n<N$, two or more of the Lamé functions for the N components may be the same function. The number of N combinations that can be chosen from $2n+1$ distinct Lamé functions of order $n<N$ each of which may appear from 0 to N times is $M'=\binom{2n+N}{N}$.

A total of M' combinations $(m_1, m_2, \ldots, m_N)_n$ are possible where $m_1 \leqslant m_2 \leqslant m_3 \leqslant \ldots \leqslant m_N$ can be chosen, with repetition allowed, from 1, 2, $2'$, 3, $3'$, $\ldots, n+1$, $(n+1)'$, and in general, these combinations are possible solutions for each of the 2^N interaction types, but with a number of exceptions, some specific and some general. The following restrictions apply generally:

(a) A combination $(m_1, m_2, \ldots, m_N)_n$ must have at least n distinct m's for it to be a solution; and

(b) combination $(m_1, m_2, \ldots, m_N)_n$ is disallowed for interaction type $((-1)^{m_1+n}, (-1)^{m_2+n}, \ldots, (-1)^{m_N+n})$.

III. COMBINATIONS AND INTERACTION TYPES

For the case of $N>n$ for which combinations with repetitions are allowed, a combination in which two or more of the m's in the combination $(m_1, m_2, \ldots, m_N)_n$ are equal is referred to as a degenerate combination, and one in which all the m's are distinct as a nondegenerate combination.

For the case of $n=N$, all combinations must be nondegenerate (for $0<k^2<1$) to be solutions of N CNLS equations (4). It is useful to divide the $M=\binom{2n+1}{N}$ combinations of Lamé functions that can be obtained from $2n+1$ Lamé functions of order n (with no repetition allowed) into two kinds: the degenerative combination is one in which at least two of the m's in the combination $(m_1, m_2, \ldots, m_N)_n$ involve a pair p and p', where p is one of the numbers from 2 to $n+1$; the nondegenerative combination is one that is not degenerative. Of the M combinations, the number of degenerative combinations is $M_d = n(2n-1)!/\{(N-2)!(2n-N+1)!\}$ and the number of nondegenerative combinations is $M_{nd} = M - M_d$.

The 2^N interaction types are divided into two subsets: (1) the interaction types in which all the $-$'s precede the $+$'s belong to the "weakly" mixed type or subset that includes the "pure" type in which the signs are all $-$'s or all $+$'s, as two special cases; and (2) the interaction types in which the $-$'s that appear before *and* after the $+$'s belong to the "strongly" mixed type or subset. The weakly mixed subset consists of $N+1$ interaction types, and the strongly mixed subset consists of the remaining $2^N - N - 1$ interaction types.

The nondegenerative combinations are found to be solutions of only the weakly mixed interaction type. The degenerative combinations can be solutions of the strongly mixed as well as the weakly mixed interaction types excepting the pure type. Division into degenerative and nondegenerative combinations aside, every one of the M possible combinations (for $n=N$) of Lamé functions is a solution to one and only one of the 2^N interaction types.

In Appendix B, we present, for $N=1$–3, the combinations of Lamé functions that give analytic solutions for each of the 2^N interaction types, the Lamé functions being those given in Appendix A. The grouping can be verified to be in agreement with the rules given above. To complete the analytic solutions, we need the explicit expressions for the amplitudes C's for Eq. (8) and the required c's for Eq. (4). Some solutions given previously[6,11–13] were for the pure type only. For $N=1$ and 2, the solutions for the entire set were given in Ref. 14. For $N=3$, the number of possible combinations for $n=3$ alone is 35, and there are many combinations that are solutions for $n=1$ and 2. We present, in Appendix C, the complete analytic solutions for five specific combinations, pure and mixed, with examples from $n=1$, 2 and 3, for which the Lamé functions given in Appendix A are used and Eq. (8) applies.

The complete analytic solutions for $N=3$ for the case of $k^2=1$, for which the $2n+1$ Lamé functions of order n coalesce into $n+1$ associated Legendre functions of order n, will be presented later. Indeed, for the case $k^2=1$, two special cases allow very compact expressions for a general N which we shall discuss in the next section.

The division of combinations into degenerative and nondegenerative is consistent with consideration of Lamé functions as k^2 becomes equal to 1. In that case, the $2n+1$ eigenvalues, except for the first one, become pairwise degenerate, i.e., $h_2^{(n)} = h_{2'}^{(n)}, h_3^{(n)} = h_{3'}^{(n)}, \ldots, h_{n+1}^{(n)} = h_{(n+1)'}^{(n)}$, and the corresponding Lamé functions of order n become $n+1$ associated Legendre functions of order n, $P_n^m(x)$, $m=0,1,2,\ldots,n$, where $x=\tanh(\alpha t)$. These (unnormalized) Legendre functions are presented in Appendix D.

IV. ANALYTIC SOLUTIONS FOR $k^2=1$

We shall present in this section all analytic solutions for $N=1$–3 for $k^2=1$ for all 2^N interaction types. For the weakly mixed interaction type, we have compact closed-form expressions of analytic solutions for a general value of N for two special cases that are derived from studying the equations obtained with the use of the ansatz.[5]

We first define our normalized associated Legendre functions $P_n^m(x)$ as follows:

$$P_n^m(x) = (1-x^2)^{m/2} d^m P_n(x)/dx^m,$$

where $P_n(x) = (2^{-n}/n!) d^n (x^2-1)^n/dx^n$.

Consider the following $(N+1)$ CNLS equations involving ψ_j, $j=1,2,\ldots,N+1$:

$$\psi_{mtt} + c_m \psi_m + \left(\sum_{j=1}^{s} -\psi_j^2 + \sum_{j=s+1}^{N+1} \psi_j^2 \right) \psi_m = 0, \quad m=1,\ldots,N+1, \tag{9}$$

where s can take the value from 0 to $N+1$. It can be verified that the following analytic solution for ψ_j satisfies Eq. (9):

$$\psi_j = C_j P_N^{j-1}(\tanh \alpha t), \quad j = 1,\ldots,N+1, \tag{10}$$

$$C_1 = \pm c_1, \quad + \text{for } s \geqslant 1, \quad - \text{for } s = 0,$$

$$C_j = \pm\{[(N-j+1)!][c_1 - 2(j-1)^2\alpha^2]/(N+j-1)!\}^{1/2},$$

where $+$ for $j=2,\ldots,s$, $s \geqslant 2$, $-$ for $j = s+1,\ldots,N+1$, $N \geqslant s$,

$$c_j = c_1 - (j-1)^2\alpha^2, \quad j = 1,\ldots,N+1,$$

$$c_1 < 0 \quad \text{for} \quad s = 0,$$

$$2s^2\alpha^2 > c_1 > 2(s-1)^2\alpha^2 \quad \text{for} \quad N+1 > s \geqslant 1,$$

$$c_1 > 2N^2\alpha^2 \quad \text{for} \quad s = N+1.$$

We now consider the above $(N+1)$ CNLS equations involving ψ_j, $j = 1,2,\ldots,N+1$ with $j = r$ missing (or $\psi_r = 0$), where r can be any one of the $N+1$ components. Written explicitly, we have the following equations:

$$\psi_{mtt} + c_m\psi_m + \left(\sum_{j=1}^{r-1} -\psi_j^2 + \sum_{j=r+1}^{N+1} \psi_j^2\right)\psi_m = 0, \quad m = 1,\ldots,N+1. \tag{11}$$

We find the following analytic solutions ψ_j with amplitudes C_j and the required c_j given by

$$\psi_j = C_j P_N^{j-1}(\tanh \alpha t), \quad j = 1,\ldots,N+1, \quad j \neq r, \tag{12}$$

$$C_1 = [2(r-1)^2]^{1/2}\alpha,$$

$$C_j = \{4[(N-j+1)!]|(r-1)^2 - (j-1)^2|/(N+j-1)!\}^{1/2}\alpha, \quad \text{for} \quad j > 1,$$

$$c_j = [2(r-1)^2 - (j-1)^2]\alpha^2, \quad j = 1,\ldots,N+1, \quad j \neq r,$$

where the specification $j \neq r$ is not really necessary since setting $j = r$ would give C_j and c_j equal to zero anyway. However, the specification $j \neq r$ is a reminder that there are N and not N+1 coupled components of CNLS equations considered.

Equation (12) gives a compact expression for the analytic solution for the L-set of N CNLS equations for the weakly mixed interaction type in terms of a combination of (different) associated Legendre functions of order $n = N$. It can be checked that for $r = 1$, i.e., when the nonlinear coupling parameters in Eq. (11) are all equal to $+1$, the above results coincide with those given previously in Refs. 4–6, and for $r = N+1$, i.e., when the nonlinear coupling parameters are all equal to -1, the above results coincide with those given previously in Ref. 12. Thus our above results given by Eq. (12) generalize the previous results to the weakly mixed type where the first $r-1$ of the β's in Eqs. (11) are equal to -1, and the remaining β's are equal to $+1$. It may be noted that the "generalized" dark solitary wave is $P_n^0(\tanh \alpha t)$ and the "generalized" bright solitary wave is $P_n^n(\tanh \alpha t)$; they become the familiar dark solitary wave $\tanh \alpha t$ and the familiar bright solitary wave $\operatorname{sech} \alpha t$, respectively, for $n = 1$. The generalized dark solitary waves $P_n^0(\tanh \alpha t)$, unlike the rest of the set $P_n^m(\tanh \alpha t)$, $m = 1,\ldots,n$, do not become zero when $t \to \pm\infty$.

By replacing N by $N-1$, Eq. (10) can be seen to provide a compact expression for a particular analytic solution for the L-set of N CNLS equations for the weakly mixed interaction type in terms of N different associated Legendre functions of order $n = N-1$. Unlike the analytic solution given by Eq. (12) for Eq. (11) which is the only analytic solution in terms of associated Legendre functions of order $n=N$, there is generally more than one analytic solution in terms of Legendre functions of order $n=N-1$ for $n<N$ generally: the other possibilities include combinations in which the same Legendre functions represent different components.

For the strongly mixed interaction type, the ansatz and procedure given in Ref. 5 are used to obtain the analytic solutions. The complete analytic solutions for $N=1-3$ in terms of associated Legendre functions of order $n\leqslant N$ are presented in Appendix E for all 2^N interaction types. To use these results, the solution of Eq. (4) for N CNLS equations characterized by the interaction type $(\beta_1,\ldots,\beta_N)$ for combination $(p_1,\ldots,p_N)$ is expressed in the form

$$\psi_m(t)=C_m f_{p_m}^{(n)}(x), \quad m=1,\ldots,N, \tag{13}$$

where the associated Legendre functions f's used are those given in Appendix D, the C's for Eq. (13) and the required c's for Eq. (4) are given in Appendix E, and where $x\equiv\tanh(\alpha t)$, α being a scaling parameter. Note that the $f_q^{(n)}(x)$ used for Eq. (13) differs from the normalized associated Legendre function $P_n^{(q-1)}(x)$ used for Eqs. (10) and (12) by a normalization constant. The solutions for $\psi_m(t)$ obtained from Eqs. (13) must of course be the same as those obtained from Eqs. (10) or (12) for the corresponding cases. Notice that combinations of associated Legendre functions of order $n=N$ are absent for the strongly mixed interaction type for $k^2=1$, which can be understood from the rules described in Secs. II and III (see also Appendix B for $0<k^2<1$).

The important role played by coupling a wave to another wave or to two other waves can now be seen in full view in Appendix E. For example, the wave $f_1^{(1)}$ that can by itself be a solitary wave only in the normal GVD region $[(1)_1$ in $(-)]$ can be a solitary wave in the anomalous GVD region if it is coupled with another wave $f_2^{(1)}$ $[(1,2)_1$ in $(++)]$; and the coupled waves $f_1^{(2)}$ and $f_2^{(2)}$ that can be a solitary-wave pair only in the normal GVD region $[(1,2)_2$ in $(--)]$ can be made into solitary waves in the anomalous GVD region if the pair is coupled to a third wave $f_3^{(2)}$ $[(1,2,3)_2$ in $(+++)]$.

V. SUMMARY

We have reviewed the L-set for N CNLS equations that possess analytic solutions in terms of N combinations of Lamé functions of order $n\leqslant N$ for 2^N interaction types. That every combination of Lamé functions of order $n=N$ is a solution for *one and only one* interaction type, and that every interaction type has *at least one* combination of Lamé functions of order $n=N$ as solution, as were first presented in Ref. 8, clearly tie the Lamé functions of order $n=N$ very intimately with the L-set for N CNLS equations, and may indicate a relationship of a deeper significance to be uncovered. The relationship between the weakly mixed interaction type and the nondegenerative combinations of Lamé functions order $n=N$ is another curiously interesting result.

For $k^2=1$, the weakly mixed interaction type allows the analytic solution in a combination of associated Legendre functions of order $n=N$ to be expressed in a compact closed-form expression for a general N, and likewise for a combination of associated Legendre functions of order $n=N-1$. Generally for $0<k^2\leqslant 1$ and every possible interaction type, however, the analytic solutions must be obtained individually, for which the ansatz and procedure prescribed in Ref. 5 can be used efficiently.

The complete analytic solutions presented for $N=1-3$ in Appendix E for $k^2=1$, for combinations of associated Legendre functions of order $n\leqslant N$, and for all 2^N interaction types, provide an excellent overview of the important role played by nonlinear couplings. For example, it enables one to easily see that coupling another wave or two other waves to a given wave can make all partners propagate as solitary waves in certain GVD regions when the individual waves cannot propagate as solitary waves by themselves. The robustness of the combinations must await experimental testing which hopefully will be used also to discover the required general pattern for certain combinations to be robust and practically useful. There is another useful application of our results. As presented in a recent communication,[10] they provide the analytic forms of stationary distributions of coupled Bose–Einstein condensates that can be used to suggest possible ways of making condensates overlap each other (i.e., same functions appearing in the combination) or separated (i.e. different functions in the combination). The mixed interaction types, which make the L-set complete in terms of its relationship to Lamé functions of order $n=N$, are clearly important mathematically, but their physical applications still await experimental explorations.

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APPENDIX A: LAMÉ FUNCTIONS FOR $n=1-3$

The $2n+1$ Lamé functions $f_m^{(n)}(\tau)$ and their eigenvalues $h_m^{(n)}$ satisfy the Lamé equations (7), and we list them for $n=1-3$ according to the subscript $m=1,2,2',3,3',\ldots,n+1,(n+1)'$ in the following.

1. $n=1$

$$h_1^{(1)}=1+k^2,\quad h_2^{(1)}=1,\quad h_{2'}^{(1)}=k^2,$$

$$f_1^{(1)}=sn(\tau),\quad f_2^{(1)}=cn(\tau),\quad f_{2'}^{(1)}=dn(\tau).$$

2. $n=2$

$$h_{1,3'}^{(2)}=2(1+k^2)\pm 2\sqrt{1-k^2+k^4},$$

$$h_2^{(2)}=4+k^2,\quad h_{2'}^{(2)}=1+4k^2,\quad h_3^{(2)}=1+k^2,$$

$$f_{1,3'}^{(2)}=\frac{1}{3}(1+k^2\mp\sqrt{1-k^2+k^4})-k^2sn^2(\tau),$$

$$f_2^{(2)}=sn(\tau)cn(\tau),\quad f_{2'}^{(2)}=sn(\tau)dn(\tau),\quad f_3^{(2)}=cn(\tau)dn(\tau).$$

3. $n=3$

$$h_{1,3'}^{(3)}=5(1+k^2)\pm 2\sqrt{4-7k^2+4k^4},$$

$$h_{2,4}^{(3)}=5+2k^2\pm 2\sqrt{4-k^2+k^4},$$

$$h_{2',4'}^{(3)}=2+5k^2\pm 2\sqrt{1-k^2+4k^4},$$

$$h_3^{(3)}=4(1+k^2),$$

$$f_{1,3'}^{(3)}=sn(\tau)\{1-a_{1,3'}sn^2(\tau)\},$$

$$f_{2,4}^{(3)}=cn(\tau)\{1-a_{2,4}sn^2(\tau)\},$$

$$f_{2',4'}^{(3)}=dn(\tau)\{1-a_{2',4'}sn^2(\tau)\},$$

$$f_3^{(3)}=sn(\tau)cn(\tau)dn(\tau),$$

where

$$a_{1,3'} = \frac{1}{3}[2 + 2k^2 \pm \sqrt{4 - 7k^2 + 4k^4}],$$

$$a_{2,4} = 2 + k^2 \pm \sqrt{4 - k^2 + k^4},$$

$$a_{2',4'} = 1 + 2k^2 \pm \sqrt{1 - k^2 + 4k^4}.$$

APPENDIX B: COMBINATIONS AND INTERACTION TYPES FOR $N=1-3$

A collection of N Lamé functions $(f^{(n)}_{m_1}, f^{(n)}_{m_2}, \ldots, f^{(n)}_{m_N})$ or $(m_1, m_2, \ldots, m_N)_n$ chosen from $2n+1$ Lamé functions $f^{(n)}_m$ of order n that can serve as an analytic solution for the N components ψ_m, $m=1,\ldots,N$, of Eqs. (4) is referred to as a combination. In this Appendix, we list, for every one of the 2^N possible interaction types $(\beta_1, \beta_2, \ldots, \beta_N)$, where β_j can be $+1$ or -1, for Eq. (4), all the possible combinations for Lamé functions of order $n=N$, for $N=1-3$ [the subscript n in the combination $(m_1, m_2, \ldots, m_N)_n$ is dropped as it is understood that $n=N$]. The total number M of possible combinations for $N=1,2,3$ are 3, 10, and 35, respectively.

We list only the "principal" combinations with the number of total possible combinations that can be obtained from them by changing, say, 2 to $2'$, 3 to $3'$, etc., given in the square brackets that follow, remembering the restriction that for any combination $(m_1, m_2, \ldots, m_N)_n$ for the case $n=N$, $m_1 < m_2 < \ldots < m_N$. For example, (2)[2] represents two combinations $(2)_1$ and $(2')_1$, and (1,2,3)[4] represents four combinations $(1,2,3)_3$, $(1,2,3')_3$, $(1,2',3)_3$ and $(1,2',3')_3$.

Interaction type	Combination
$N=1$	
$(-)$	(1)[1]
$(+)$	(2)[2]
$N=2$	
$(--)$	(1,2)[2]
$(-+)$	(1,3)[2],(3,3′)[1]
$(+-)$	(2,2′)[1]
$(++)$	(2,3)[4]
$N=3$	
$(---)$	(1,2,3)[4]
$(--+)$	(1,2,2′)[1],(1,2,4)[4],(1,4,4′)[1], (3,4,4′)[2]
$(-+-)$	(1,3,3′)[1]
$(-++)$	(1,3,4)[4],(3,3′,4)[2]
$(+--)$	(2,2′,3)[2]
$(+-+)$	(2,2′,4)[2],(2,4,4′)[2]
$(++-)$	(2,3,3′)[2]
$(+++)$	(2,3,4)[8]

APPENDIX C: SOLUTIONS FOR FIVE COMBINATIONS FOR $N=3$

In this Appendix, we give some representative examples of analytic solutions for $N=3$ of Eq. (4) for various interaction types: three combinations consist of Lamé functions of order 3, and one combination each consists of Lamé functions of orders 1 and 2. For combination $(p_1, p_2, p_3)_n$, the analytic solution is $\psi_m = C_m f^{(n)}_{p_m}$, $m=1,2,3$, where f's are given in Appendix A, and C's and the required c's for Eq. (4) are given in this Appendix.

1. $N=3$

(*i*) Combination $(1,2,3)_3$ for interaction type $(---)$.

$$C_j=(\Delta_j/\Delta)^{1/2},$$

where

$$\Delta_1=108k^2\alpha^2(8-k^2+4\sqrt{4-k^2+k^4}),$$

$$\Delta_2=12k^2\alpha^2\{8-3k^2-3k^4+8k^6+2(2+k^2+2k^4)\sqrt{4-7k^2+4k^4}\},$$

$$\Delta_3=12k^4\alpha^2\{-80+57k^2+18k^4+16k^6+(-50+26k^2+16k^4)\sqrt{4-k^2+k^4}+(50+16k^2 +8k^4)\sqrt{4-7k^2+4k^4}+(20+8k^2)\sqrt{(4-7k^2+4k^4)(4-k^2+k^4)}\},$$

$$\Delta=32-90k^2+78k^4-16k^6+4(5-11k^2+8k^4)\sqrt{4-k^2+k^4}+2(-10+16k^2 -4k^4)\sqrt{4-7k^2+4k^4}+8(-1+2k^2)\sqrt{(4-7k^2+4k^4)(4-k^2+k^4)},$$

$$c_j=\Delta_j/\Delta+h_j^{(3)}\alpha^2.$$

(*ii*) Combination $(1,3,3')_3$ for interaction type $(-+-)$.

$$C_{1,3}=\sqrt{3}k'^{-2}k\alpha\left(2+k^2+2k^4\mp\frac{(1+k^2)(8-11k^2+8k^4)}{2\sqrt{4-7k^2+4k^4}}\right)^{1/2},$$

$$C_2=\sqrt{30}k'^{-2}k\alpha,$$

$$c_{1,3}=\{5(1+k^2)\pm2\sqrt{4-7k^2+4k^4}\}\alpha^2,\quad c_2=4(1+k^2)\alpha^2.$$

(*iii*) Combination $(2,3,4)_3$ for interaction type $(+++)$.

$$C_{1,3}=\sqrt{3}k\alpha\left(2\mp\frac{8-k^2}{2\sqrt{4-k^2+k^4}}\right)^{1/2},\quad C_2=\sqrt{30}k^2\alpha,$$

$$c_{1,3}=\{5(1-2k^2)\pm2\sqrt{4-k^2+k^4}\}\alpha^2,\quad c_2=4(1-2k^2)\alpha^2.$$

(*iv*) Combination $(1,2,3)_2$ for interaction type $(\beta_1,\beta_2,\beta_3)$, excluding $(-+-)$ and $(+-+)$.

$$C_1=(D_1/D)^{1/2},\quad C_2=(D_2/D)^{1/2}+6k^4\alpha^2,\quad C_3=(D_3/D)^{1/2}+6k^2\alpha^2,$$

where

$$D_1=9\beta_1\{-c_1+[2(1-2k^2)+2\sqrt{1-k^2+k^4}]\alpha^2\},$$

$$D_2=3\beta_2k^4(1+k^2+2\sqrt{1-k^2+k^4})\{-c_1+[2(1-2k^2)+2\sqrt{1-k^2+k^4}]\alpha^2\},$$

$$D_3=3\beta_3k^2(-2+k^2+2\sqrt{1-k^2+k^4})\{-c_1+[2(1-2k^2)+2\sqrt{1-k^2+k^4}]\alpha^2\},$$

$$D=2-5k^2+5k^4-2(1-2k^2)\sqrt{1-k^2+k^4},$$

$$c_1-c_2=-2+k^2+2\sqrt{1-k^2+k^4},\quad c_1-c_3=1+k^2+2\sqrt{1-k^2+k^4}.$$

(v) Combination $(1,2,2)_1$ for interaction type $(\beta_1,\beta_2,\beta_3)$, excluding $(+--)$.

$$C_1=\beta_1\{-c_1+(1-k^2)\alpha^2\},$$

$$\beta_2 C_2^2+\beta_3 C_3^2=-c_1+(1+k^2)\alpha^2,$$

$$c_1-c_2=k^2\alpha^2,\quad c_2=c_3,$$

$$\text{and for}\quad \beta_1=+1,\quad c_1<(1-k^2)\alpha^2,$$

$$\text{and for}\quad \beta_1=-1,\quad c_1>(1-k^2)\alpha^2.$$

APPENDIX D: ASSOCIATED LEGENDRE FUNCTIONS FOR $n=1-3$

In this Appendix, we list the $n+1$ associated Legendre functions of order $n=1-3$ that are to be used for Eq. (13) in conjunction with Appendix E.

1. $n=1$

$$f_1^{(1)}=\tanh(\alpha t)\equiv x,\quad f_2^{(1)}=\mathrm{sech}(\alpha t)=(1-x^2)^{1/2}.$$

2. $n=2$

$$f_1^{(2)}=\mathrm{sech}^2(\alpha t)-\frac{2}{3}=\frac{1}{3}-x^2,$$

$$f_2^{(2)}=\tanh(\alpha t)\mathrm{sech}(\alpha t)=x(1-x^2)^{1/2},$$

$$f_3^{(2)}=\mathrm{sech}^2(\alpha t)=1-x^2.$$

3. $n=3$

$$f_1^{(3)}=\tanh(\alpha t)\left\{\mathrm{sech}^2(\alpha t)-\frac{2}{5}\right\}=x\left(\frac{3}{5}-x^2\right),$$

$$f_2^{(3)}=\mathrm{sech}(\alpha t)\left\{\mathrm{sech}^2(\alpha t)-\frac{4}{5}\right\}=(1-x^2)^{1/2}\left(\frac{1}{5}-x^2\right),$$

$$f_3^{(3)}=\tanh(\alpha t)\mathrm{sech}^2(\alpha t)=x(1-x^2),$$

$$f_4^{(3)}=\mathrm{sech}^3(\alpha t)=(1-x^2)^{3/2}.$$

APPENDIX E: SPECIAL SOLUTIONS FOR $N=1-3$

In this Appendix, we give all analytic solutions of Eq. (4) for 2^N interaction types in terms of combinations of associated Legendre functions given in Appendix D, for $N=1-3$. See Eq. (13) for applications of the parameters given here.

1. $N=1$

$\underline{(-)}$ $(1)_1$ $C_1=\sqrt{2}\alpha$, $c_1=2\alpha^2$.

$\underline{(+)}$ $(2)_1$ $C_1=\sqrt{2}\alpha$, $c_1=-\alpha^2$.

2. $N=2$

$\underline{(--)}$

$$(1,2)_2 \quad C_1=C_2=3\sqrt{2}\alpha, \quad c_1=8\alpha^2, \quad c_2=7\alpha^2.$$

$$(1,1)_1 \quad C_1^2+C_2^2=2\alpha^2, \quad c_1=c_2=2\alpha^2.$$

$$(1,2)_1 \quad C_1^2=c_1, \quad C_2^2=c_1-2\alpha^2, \quad c_1-c_2=\alpha^2, \quad c_1>2\alpha^2.$$

$\underline{(-+)}$

$$(1,3)_2 \quad C_1=C_2=3\sqrt{2}\alpha/2, \quad c_1=2\alpha^2, \quad c_2=-2\alpha^2.$$

$$(1,1)_1 \quad C_1^2-C_2^2=2\alpha^2, \quad c_1=c_2=2\alpha^2.$$

$$(1,2)_1 \quad C_1^2=c_1, \quad C_2^2=-c_1+2\alpha^2, \quad c_1-c_2=\alpha^2, \quad 2\alpha^2>c_1>0.$$

$$(2,2)_1 \quad -C_1^2+C_2^2=2\alpha^2, \quad c_1=c_2=-\alpha^2.$$

$\underline{(+-)}$

$$(1,1)_1 \quad -C_1^2+C_2^2=2\alpha^2, \quad c_1=c_2=2\alpha^2.$$

$$(2,2)_1 \quad C_1^2-C_2^2=2\alpha^2, \quad c_1=c_2=-\alpha^2.$$

$\underline{(++)}$

$$(2,3)_2 \quad C_1=C_2=\sqrt{6}\alpha, \quad c_1=-\alpha^2, \quad c_2=-4\alpha^2.$$

$$(1,2)_1 \quad C_1^2=-c_1, \quad C_2^2=-c_1+2\alpha^2, \quad c_1-c_2=\alpha^2, \quad c_1<0.$$

3. $N=3$

$\underline{(---)}$

$$(1,2,3)_3 \quad C_1=15\sqrt{2}\alpha/2, \quad C_2=5\sqrt{6}\alpha, \quad C_3=5\sqrt{6}\alpha/2, \quad c_1=18\alpha^2, \quad c_2=17\alpha^2, \quad c_3=14\alpha^2.$$

$$(1,1,2)_2 \quad C_1^2+C_2^2=C_3^2=18\alpha^2, \quad c_1=c_2=8\alpha^2, \quad c_3=7\alpha^2.$$

$$(1,2,2)_2 \quad C_1^2=C_2^2+C_3^2=18\alpha^2, \quad c_1=8\alpha^2, \quad c_2=c_3=7\alpha^2.$$

$$(1,2,3)_2 \quad C_1^2=9c_1/4, \quad C_2^2=3(c_1-2\alpha^2), \quad C_3^2=3(c_1-8\alpha^2)/4, \quad c_1>4\alpha^2,$$

$$c_2=c_1-\alpha^2, \quad c_3=c_1-4\alpha^2.$$

$$(1,1,1)_1 \quad C_1^2+C_2^2+C_3^2=2\alpha^2, \quad c_1=c_2=c_3=2\alpha^2.$$

$$(1,1,2)_1 \quad C_1^2+C_2^2=c_1, \quad C_3^2=c_1-2\alpha^2, \quad c_1>2\alpha^2, \quad c_1=c_2, \quad c_1-c_3=\alpha^2.$$

$$(1,2,2)_1 \quad C_1^2=c_1, \quad C_2^2+C_3^2=c_1-2\alpha^2, \quad c_1>2\alpha^2.$$

$\underline{(--+)}$

$(1,2,4)_3 \quad C_1=5\sqrt{2}\alpha, \quad C_2=15\alpha/2, \quad C_3=5\alpha/2, \quad c_1=8\alpha^2, \quad c_2=7\alpha^2, \quad c_3=-\alpha^2.$

$(1,1,3)_2 \quad C_1^2+C_2^2=C_3^2=9\alpha^2/2, \quad c_1=c_2=2\alpha^2, \quad c_3=-2\alpha^2.$

$(1,2,2)_2 \quad C_1^2=C_2^2-C_3^2=18\alpha^2, \quad c_1=8\alpha^2, \quad c_2=c_3=7\alpha^2.$

$(1,2,3)_2 \quad C_1^2=9c_1/4, \quad C_2^2=3(c_1-2\alpha^2), \quad C_3^2=3(-c_1+8\alpha^2)/4, \quad 8\alpha^2>c_1>2\alpha^2,$

$c_2=c_1-\alpha^2, \quad c_3=c_1-4\alpha^2.$

$(1,3,3)_2 \quad C_1^2=-C_2^2+C_3^2=9\alpha^2/2, \quad c_1=2\alpha^2, \quad c_2=c_3=-2\alpha^2.$

$(1,1,1)_1 \quad C_1^2+C_2^2-C_3^2=2\alpha^2, \quad c_1=c_2=c_3=2\alpha^2.$

$(1,1,2)_1 \quad C_1^2+C_2^2=c_1, \quad C_3^2=-c_1+2\alpha^2, \quad 0<c_1<2\alpha^2, \quad c_1=c_2, \quad c_1-c_3=\alpha^2.$

$(1,2,2)_1 \quad C_1^2=c_1, \quad -C_2^2+C_3^2=-c_1+2\alpha^2, \quad c_1>0, \quad c_2=c_3, \quad c_1-c_2=\alpha^2.$

$(2,2,2)_1 \quad C_1^2+C_2^2-C_3^2=-2\alpha^2, \quad c_1=c_2=c_3=-\alpha^2.$

$\underline{(-+-)}$

$(1,1,2)_2 \quad C_1^2-C_2^2=C_3^2=18\alpha^2, \quad c_1=c_2=8\alpha^2, \quad c_3=7\alpha^2.$

$(1,2,2)_2 \quad C_1^2=-C_2^2+C_3^2=18\alpha^2, \quad c_1=8\alpha^2, \quad c_2=c_3=7\alpha^2.$

$(1,3,3)_2 \quad C_1^2=C_2^2-C_3^2=9\alpha^2/2, \quad c_1=2\alpha^2, \quad c_2=c_3=-2\alpha^2.$

$(1,1,1)_1 \quad C_1^2-C_2^2+C_3^2=2\alpha^2, \quad c_1=c_2=c_3=2\alpha^2.$

$(1,1,2)_1 \quad -C_1^2+C_2^2=c_1, \quad C_3^2=c_1-2\alpha^2, \quad c_1>2\alpha^2, \quad c_1=c_2, \quad c_1-c_3=\alpha^2.$

$(1,2,2)_1 \quad C_1^2=c_1, \quad C_2^2-C_3^2=-c_1+2\alpha^2, \quad c_1>0, \quad c_2=c_3, \quad c_1-c_2=\alpha^2.$

$(2,2,2)_1 \quad C_1^2-C_2^2+C_3^2=-2\alpha^2, \quad c_1=c_2=c_3=-\alpha^2.$

$\underline{(-++)}$

$(1,3,4)_3 \quad C_1=5\sqrt{2}\alpha/2, \quad C_2=3\sqrt{10}\alpha/2, \quad C_3=\sqrt{10}\alpha, \quad c_1=2\alpha^2, \quad c_2=-2\alpha^2, \quad c_3=-7\alpha^2.$

$(1,1,3)_2 \quad C_1^2-C_2^2=C_3^2=9\alpha^2/2, \quad c_1=c_2=2\alpha^2, \quad c_3=-2\alpha^2.$

$(1,2,3)_2 \quad C_1^2=9c_1/4, \quad C_2^2=3(-c_1+2\alpha^2), \quad C_3^2=3(-c_1+8\alpha^2)/4, \quad 2\alpha^2>c_1>0,$

$c_2=c_1-\alpha^2, \quad c_3=c_1-4\alpha^2.$

$(1,3,3)_2 \quad C_1^2=C_2^2+C_3^2=9\alpha^2/2, \quad c_1=2\alpha^2, \quad c_2=c_3=-2\alpha^2.$

$(2,2,3)_2 \quad -C_1^2+C_2^2=C_3^2=6\alpha^2, \quad c_1=c_2=-\alpha^2, \quad c_3=-4\alpha^2.$

$(1,1,1)_1 \quad C_1^2-C_2^2-C_3^2=2\alpha^2, \quad c_1=c_2=c_3=2\alpha^2.$

$(1,1,2)_1 \quad C_1^2-C_2^2=c_1, \quad C_3^2=-c_1+2\alpha^2, \quad c_1<2\alpha^2, \quad c_1=c_2, \quad c_1-c_3=\alpha^2.$

$(1,2,2)_1 \quad C_1^2=c_1, \quad C_2^2+C_3^2=-c_1+2\alpha^2, \quad 2\alpha^2>c_1>0, \quad c_2=c_3, \quad c_1-c_2=\alpha^2.$

$(2,2,2)_1 \quad C_1^2-C_2^2-C_3^2=-2\alpha^2, \quad c_1=c_2=c_3=-\alpha^2.$

$\underline{(+--)}$

$(1,1,2)_2 \quad -C_1^2+C_2^2=C_3^2=18\alpha^2, \quad c_1=c_2=8\alpha^2, \quad c_3=7\alpha^2.$

$(1,1,1)_1 \quad -C_1^2+C_2^2+C_3^2=2\alpha^2, \quad c_1=c_2=c_3=2\alpha^2.$

$(1,1,2)_1 \quad C_1^2-C_2^2=-c_1, \quad C_3^2=c_1-2\alpha^2, \quad c_1>2\alpha^2, \quad c_1=c_2, \quad c_1-c_3=\alpha^2.$

$(2,2,2)_1 \quad -C_1^2+C_2^2+C_3^2=-2\alpha^2, \quad c_1=c_2=c_3=-\alpha^2.$

$\underline{(+-+)}$

$(1,1,3)_2 \quad -C_1^2+C_2^2=C_3^2=9\alpha^2/2, \quad c_1=c_2=2\alpha^2, \quad c_3=-2\alpha^2.$

$(2,2,3)_2 \quad C_1^2-C_2^2=C_3^2=6\alpha^2, \quad c_1=c_2=-\alpha^2, \quad c_3=-4\alpha^2.$

$(2,3,3)_2 \quad C_1^2=-C_2^2+C_3^2=6\alpha^2, \quad c_1=-\alpha^2, \quad c_2=c_3=-4\alpha^2.$

$(1,1,1)_1 \quad -C_1^2+C_2^2-C_3^2=2\alpha^2, \quad c_1=c_2=c_3=2\alpha^2.$

$(1,1,2)_1 \quad C_1^2-C_2^2=-c_1, \quad C_3^2=-c_1+2\alpha^2, \quad c_1<2\alpha^2, \quad c_1=c_2, \quad c_1-c_3=\alpha^2.$

$(1,2,2)_1 \quad C_1^2=-c_1, \quad -C_2^2+C_3^2=-c_1+2\alpha^2, \quad c_1<0, \quad c_2=c_3, \quad c_1-c_2=\alpha^2.$

$(2,2,2)_1 \quad -C_1^2+C_2^2-C_3^2=-2\alpha^2, \quad c_1=c_2=c_3=-\alpha^2.$

$\underline{(++-)}$

$(2,2,3)_2 \quad C_1^2=C_2^2-C_3^2=6\alpha^2, \quad c_1=-\alpha^2, \quad c_2=c_3=-4\alpha^2.$

$(1,1,1)_1 \quad -C_1^2-C_2^2+C_3^2=2\alpha^2, \quad c_1=c_2=c_3=2\alpha^2.$

$(1,2,2)_1 \quad C_1^2=-c_1, \quad C_2^2-C_3^2=-c_1+2\alpha^2, \quad c_1<0, \quad c_1-c_2=\alpha^2, \quad c_1=c_3.$

$(2,2,2)_1 \quad -C_1^2-C_2^2+C_3^2=-2\alpha^2, \quad c_1=c_2=c_3=-\alpha^2.$

$\underline{(+++)}$

$(2,3,4)_3 \quad C_1=5\sqrt{3}\alpha/2, \quad C_2=\sqrt{30}\alpha, \quad C_3=3\sqrt{5}\alpha/2, \quad c_1=-\alpha^2, \quad c_2=-4\alpha^2, \quad c_3=-9\alpha^2.$

$(1,2,3)_2 \quad C_1^2=-9c_1/4, \quad C_2^2=3(-c_1+2\alpha^2), \quad C_3^2=3(-c_1+8\alpha^2)/4,$

$c_1<0, \quad c_2=c_1-\alpha^2, \quad c_3=c_1-4\alpha^2.$

$(2,2,3)_2 \quad C_1^2+C_2^2=C_3^2=6\alpha^2, \quad c_1=c_2=-\alpha^2, \quad c_3=-4\alpha^2.$

$(2,3,3)_2 \quad C_1^2=C_2^2+C_3^2=6\alpha^2, \quad c_1=-\alpha^2, \quad c_2=c_3=-4\alpha^2.$

$$(1{,}1{,}2)_1 \quad C_1^2+C_2^2=-c_1, \quad C_3^2=-c_1+2\alpha^2, \quad c_1<0, \quad c_1=c_2, \quad c_1-c_3=\alpha^2.$$

$$(1{,}2{,}2)_1 \quad C_1^2=-c_1, \quad C_2^2+C_3^2=-c_1+2\alpha^2, \quad c_1<0, \quad c_2=c_3, \quad c_1-c_2=\alpha^2.$$

$$(2{,}2{,}2)_1 \quad C_1^2+C_2^2+C_3^2=2\alpha^2, \quad c_1=c_2=c_3=-\alpha^2.$$

[1] See, e.g., G. P. Agrawal, *Nonlinear Fiber Optics*, 3rd ed. (Academic, New York, 2001), Chap. 7; N. Akhmediev and A. Ankiewicz, *Solitons: Nonlinear Pulses and Beams* (Chapman and Hall, London, 1997).
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