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Questioning Strategies that Promote Critical Thinking

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Questioning Strategies that Promote Critical Thinking
Deborah Horowitz
St. John Fisher College
Abstract

Teachers use questions in their respective classroom on a daily basis as a part of their instructional practices. Research into questioning strategies and sequences indicated that the incorporation of purposeful and effective questions, variety of techniques, and the sequence of questions had a significant impact on both teachers and students. Results indicated that teachers were more aware of what and how the students were learning and thinking. Teachers became more familiar with their content and the expectations of their students, had more meaningful and frequent communication with each other, as well as between students. Students became better problem solvers, validated their own responses, and pursued alternatives in solutions. Teachers and students became more vested in the learning process.
Dedication

To my family for all their support, understanding, and patience for the past five years since I went back into teaching and began my own inquiry based learning of the new teaching pedagogy and continued my education as a student and an educator.

To my research team for being great colleagues to collaborate with to further not only our students learning and thinking capabilities, but our own as well and for being great friends with endless support throughout the process.

To my students for which I am forever grateful that I have been able to touch their lives and them mine and for being the source of inspiration for working on an incredible journey on how to be an effective teacher so that I may have been able to achieve my goal as a teacher, to make a difference.
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Questioning Strategies that Promote Critical Thinking

A teacher has many different roles and responsibilities that must be met in order to provide for students' learning of a particular content in his/her classroom. How do teachers know if the students are learning? How do teachers know what the students are learning? What information or knowledge do the students bring with them to the classroom? Have students retained or developed misinformation or alternative understandings from past experiences? How do the teachers know to what extent his/her students' level of understanding is? How do teachers know what students are thinking? How do teachers build on the existing knowledge of students in order to their critical thinking skills? While written assessments and other learner-developed products can provide answers to some of these questions including new knowledge that has been constructed or existing knowledge that has been augmented in varying degrees, overt indications of knowledge construction require an analysis of process data (King, 1994). Since knowledge construction is an internal cognitive process, research in this area must look for external indicators that knowledge construction is taking place and to what extent (King). One way to obtain the external indicators is to incorporate various questioning strategies or techniques and sequences in various contexts for both the teacher and the student. Questions raise new ideas and suggestions, which stimulate student thought and action while revealing a particular strand of problem-solving logic (Penick, Crow, & Bonnestetter, 1996). As Cathleen Galas remarked in 1999, "a good question leads to more new questions, new discoveries, new realms never even considered before" (p. 11).
Recognizing the importance of questions in teaching, researchers still do not know much about questioning and the impact questioning has on teaching and learning. The incorporation of questioning by both teacher and student in a classroom environment raise several questions in and of itself. What educational objectives can questions help students achieve (Gall, 1970)? What types of questions should be asked by the teacher and the student (Commeyras, 1995; Gall)? When should the questions be asked and for what purpose? What are the criteria of an effective question and how can effective questions be identified (Gall)? How do you get the students to ask questions that are not only more numerous, but effective and to the level of critical thinking? Should the questions teachers and students ask be categorized into levels of taxonomy to gauge the types of questions being asked, as well as the frequency of questions? How do teachers and students develop questioning skills? Does questioning affect the behavior of the teacher and/or student? How do students’ responses impact what questions teachers ask and how a teacher responds? How can teacher’s questioning-framing skills be improved (Gall)?

This research topic was also selected to provide guidance and an opportunity for enhancement as part of an ongoing action research that is currently being conducted by a collaborative team of educators at Penfield High School. While this research paper may not be able to answer all the questions herein, or answer some of the questions completely, the paper will highlight many of these items addressed by respected authors in the field of education. It will also serve as a more pedagogical and best-practice foundation for the continued action research of the Penfield High School collaborative team.
This research paper explored the importance of questioning in the classroom, the shift in educational paradigm in the area of mathematics that supports the need for more questioning on the part of both teacher and student (National Council of Teachers of Mathematics (NCTM), 2000; NCTM 1991; NCTM, 1989), various methods for analyzing current questioning practices and various questioning sequences. With regards to the analysis of current questioning, the research discussed whether or not the types of questions asked should be categorized and if so how. The questioning sequences that were addressed included teacher-student, student-teacher-student and student-student, along with the impact the questioning sequences had on both teacher and student behaviors. The remainder of topics addressed included other influences on how much and what type of questioning are executed by teachers in the classroom, how questioning practices, for both teacher and student, could be improved and finally what still remains to be done in the area of questioning in the classroom.

The methodology for the research consisted of four lesson studies with a collaborative team of three high school math teachers. The three teachers alternated observing and scripting the questioning of both teacher and student and their respective responses in the three classroom settings. Student samples of work associated with specific problems and questions were obtained from in-class work, homework, tickets-out-the-door, and formal and informal assessments, and small group and class discussions, where the identity of the students involved will be anonymous. Additional components of the research included comments made by teachers during debriefing and/or planning meetings and person reflections following the completion of the various lessons.
Past research on questioning strategies has indicated that, when properly sequenced, instructional questions not only foster student engagement, but the development of complex levels of thinking (Hamblen, 1984). Good questions provoke thought, are based in students’ experiences, and call for creative thinking (King, 1994). This thought and creative thinking should be demonstrated on the part of both teacher and student.

The intent of this research was to demonstrate that questioning by both teachers and students, and the analysis of the questions and associated responses led to the improvement of questioning skills by both the teacher and students which in turn led to the improvement and added success of the teachers’ instruction and students’ learning, as well as a higher level of thinking for the students in the three classrooms involved. While the study did not cover a long length of time, the research also evaluated if the questioning conducted by the teachers and students improved or moved students forward on their path of development of critical or higher-level thinking skills.
Literature Review

A fairly new topic of research in student learning has been the art of questioning by teachers and/or students and the impact the questioning has on instruction and learning in the classroom, as well as the development of students' critical thinking skills. The literature reviewed discussed various questioning strategies and the impact on students and teachers on many levels.

Typically, most classroom instruction has been monopolized by teachers employing a variety of strategies that dominate the speaking floor. make frequent request of low-level factual or recall of information, and a disregard for students hid to change the current topic (Carlsen, 1991). Additionally, prior studies in science classrooms reveal that the rate of teacher questions were dependent on the type of activity and teacher knowledge (Carlsen, 1991). The questioning rate was highest during lectures and lowest during routine seat work, with the rate of questioning having a negative correlation with teacher knowledge of the specific content.

Within the field of mathematics education, the National Council of Teachers of Teachers in Mathematics (NCTM) proposed reform (NCTM, 1989) and subsequently released standards and principles (NCTM, 2000) for teachers of mathematics which would require teachers to develop instruction activities and provide for a learning environment that encourages their students' mathematical inquiry, understanding and sense making. Thus requiring teachers to be develop strategies for complying with the new reform in mathematics education, some of which were addressed in the literature reviewed. In 1995, Michelle Commeyras stated, “creating opportunities and encouraging student-centered questioning class for a teacher-student dynamic different
from that observed in most classrooms" (p. 101). Goals which would have the teacher pose questions and create situations that allow the student to identify and investigate problems remain, for the most part, prescriptive rather than descriptive of much classroom practice (Hamblen, 1984).

While there is not a great deal of research on teacher questioning, early research on this topic has been addressed through process-product research. Process-product research has contributed many findings that helped to understand teacher questioning, however, much of the research has been focused on the effects of longer wait time and an increased use of higher-order thinking questions (Roth, 1966). This type of research does not address some more important issues, which have resulted in interpretive and sociolinguistic approaches to research into teacher questioning (Carlsen, 1988). In 1970, Meredith Gall, a significant figure in the field of questioning in education concluded that “it would be of interest to investigate the types of questions students ask [but] the more important question is to identify the types of questions which students should be encouraged to ask” (p. 716).

The literature reviewed will demonstrate how the process-product and sociolinguistic approaches to research differ, identify differing opinions as to the treatment of questions asked in the classroom, various strategies or techniques into questioning and questioning sequence such as the reflective toss (Van Zee & Minstrell, 1997), reciprocal questioning and guided peer or cooperative-questioning (King, 1990, 1994). Additionally, the extent that various questioning analysis has on the development of student's critical thinking skills, the impact on teacher and student behavior as a result of questioning and other influences on questioning in the classroom such as gender and
teacher beliefs will also be explored. The studies highlighted in the literature focused primarily on spoken questions which occurred during regular classroom teaching.

Background

Within the area of mathematics education, current reform has included discussions of and inquiry into the nature of mathematics, mathematics learning and mathematics teaching. While reform has been shaped by a number of influences, the consensus of reform has been represented by the NCTM's standards document *The Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) defined the goals for reform as all students should:

1. learn to value mathematics,
2. become confident in their ability to do mathematics,
3. become mathematical problem solvers,
4. learn to communicate mathematically,
5. learn to reason mathematically (p. 5)

The standards document promote an instructional and learning environment in which students engage in the exploration of mathematical situations, oral and written communication of ideas and modification and validation of those ideas. In a subsequent NCTM document entitled *Principles & Standards for School Mathematics* (NCTM, 2000), decisions were made by teachers, school administrators, and other education professionals which constitute a vision to guide educators as they strive for the continual improvement of mathematics education in classrooms.

Six principles for school mathematics address overarching themes that could apply to any content area:
1. Equity
2. Curriculum
3. Teaching
4. Learning
5. Assessment
6. Technology (p. 11)

The areas of Teaching and Learning have the most relevance to the research explored in this manuscript.

In the area of teaching, effective mathematics teaching requires an understanding of what students know and need to learn, as well as providing challenges and support for the students to learn. With regards to learning, students must learn mathematics with understanding, where they actively build new knowledge from experience and prior knowledge (NCTM, 1991; NCTM 2000; King, 1990). New learning is extended through student engagement situations that require an extension of their understanding and build additional connections to it (Simon, 1994). Guided high-level questioning and responding result in group members thinking about the material in new ways, as they would be confronted with differing peer perspectives on the content being studied that would be reconciled (King, 1990).

The NCTM *Principles & Standards for School Mathematics* (NCTM, 2000) identified ten standards which describe what mathematics instruction should enable students to know and do. The Content Standards explicitly describe the content that students should learn and are categorized as: “Number and Operations, Algebra, Geometry, Measurement, and Data Analysis and Probability. The Process Standards
highlight ways of acquiring and using content knowledge and are: Problem-solving, Reasoning and Proof. Communication, Connections, and Representation" (p. 30).

The categories, Data Analysis and Probability, Communication, Problem-solving, and Reasoning and Proof, require students to formulate questions, develop and evaluate inferences, make conjectures and predictions. As the standards state (NCTM, 2000) "teachers play an important role in helping to enable the development of reflective habits of mind by asking questions" (p. 54) and "by anticipating student's questions, teachers can decide if particular problems will help to further mathematical goals for the class" (p. 52). Questions such as "Why do you think this is true?" and "Does anyone think the answer is different and why do you think so?" (p. 56) help students recognize that statements need to be justified or refuted by evidence, thus engaging students in evaluating a proposed method for themselves (Van Zee & Minstrell, 1997). NCTM (2000) stated that "Listening to others' explanations gives students opportunities to develop their own understandings" and "Conversations in which mathematical ideas are explored from multiple perspectives help the participants sharpen their thinking and make connections" (p. 59). The participants in this case are both the teacher and students in the classroom. Students gain insights into their thinking when they present their algorithms for solving problems, when they justify their reasoning to a peer or teacher or when they formulate a question about something they are inquiring more knowledge on or are confused about. This is also when alternative ideas or misconceptions can be identified, explored and altered.
In order for teachers to provide instruction that reflects the shifts in the resulting reform dictated by NCTM (2000), Martin Simon in 1994 outlined areas in which teachers must develop their learning into six key components:

1. knowledge of mathematics
2. knowledge about mathematics
3. useful and personally meaningful theories of mathematics learning
4. knowledge of student development of particular mathematical ideas
5. the ability to plan instruction that adheres to the reformed principles and standards
6. the ability to interact effectively with students (i.e., listening, questioning, monitoring, and facilitating classroom discourse (p. 72)

In addition, students must learn to question and explore each other's thinking in order to clarify underdeveloped ideas that are uncovered as a result of classroom discourse. The reform documents recommend that mathematics students should be discussing and questioning their own thinking and the thinking of others (Nicol, 1998-1999). As Martin Simon stated in 1994, "Individuals involved in learning mathematics are also members of mathematical communities which are involved in the development of shared mathematical perspectives (p. 74). When students are held responsible to negotiate the difference in thinking out loud, students benefit from the discussion, while the teacher can monitor their learning (Lampert, 1990)."

**Purpose of Questioning**

The use of questioning in a classroom is an important aspect of teacher's daily practices. Researcher in the last half of a century have indicated that questioning
strategies are essential to the growth of critical thinking skills, creativity, and higher thinking skills (Marzano, 1993; Shaunessy, 2000). Questioning has many purposes which include, but are not limited to, launching a lesson or discussion, assessment of prior knowledge, student engagement, motivation (student interest) (Lampert, 1990), generate a student-centered environment, develop questioning skills (Galas, 1999; Roth, 1996), promote critical thinking skills (Hall & Myers, 1977), help students develop their own knowledge, extend knowledge (Roth, 1996), promote student accountability for the validity of their work and other classmates (Lampert, 1990; Simon & Schifter, 1991; Van Zee & Minstrell, 1997), help students clarify their meanings, provide for consideration of alternate views, monitor discussions and students thinking, influence or guide student’s thinking (Nicol, 1998-1999; Simon & Schifter, 1991, Penick et. al, 1996; Sitko & Slemon, 1982; Van Zee & Minstrell, 1997), promote an abstraction of ideas from the context of specific problems (Simon, 1994) and cognitive-memory or factual-recall (Sitko & Slemen, 1982). Most research spanning more than half a century has indicated that teachers’ questioning has been for factual recall. Traditional instruction has included questioning for the sole purpose of determining if the student has a correct response, whereby the correct responds was predetermined. The questions have not sought information but answers to be assessed in what the teacher already knows (Roth, 1996). This is due in part to the lack of effective teacher training programs (Gall, 1970).

In order to make questioning more meaningful, teachers should have the purpose for questioning in mind when formulating the types of questions to be asked, the timing of the questions, and the sequence of questioning (Penick et al., 1996; Van Zee & Minstrell, 1997). Other factors that influence teacher questioning are discussed later.
Context and Content of Questioning

William Carlsen conducted a study on *Questioning in Classroom: A Sociolinguistic Perspective* (1991), whereby he found that "the meaning of questions is dependent on the context in discourse, the content of the questions cannot be ignored and that questions reflect and sustain status differences in the classroom" (p. 157). Research on questioning has typically not addressed that classroom questions are not solely reliant on the behavior of the teacher, but are integrated forms of communication between teacher and student that are essential to instruction and learning. Carlsen also stated that past research in the form of process-product only addressed structuring, soliciting, responding and reacting as separate variables (Carlsen, 1991). However, process-product research has provided for valuable information on the positive effects of longer wait time, higher frequency of turn taking or the differential use of high-inference questions (Roth, 1996).

Carlsen's sociolinguistic approach to research began to develop descriptions and obtain results for how speakers interact in social settings within the classroom, which is imperative in student-centered, inquiry based learning environments under new mathematical reform. Context within the framework of sociolinguistic research conducted by Carlsen included detailed descriptions of the utterances by different speakers sequenced together in discourse. More specifically, Carlsen's 1991 research provided for detailed transcripts of teacher, student and other class members interactions which outline how a teacher's initial question, followed by more questions and student verbal and nonverbal responses define the context for future questions by the teacher.
With regard to content of questions, process-product research has not resulted in consistent findings. Sociolinguistic research has demonstrated that the use of certain sequences of questions help teachers maintain tight control of a discourse topic. Other research conducted in this regard has highlighted that teachers with a more extensive amount of higher knowledge planned to ask questions about material not covered in the textbook and required students to synthesize the material, teachers with lower-level of the knowledge planned to ask only recall questions (Carlsen, 1991). Content of questions could be better analyzed if a typology or classification schema is created and used for analysis and evaluation that could serve to integrate various topics and concepts (Roth, 1996).

**Questioning Strategies**

Questioning strategies most likely have an embedded purpose and/or content and context, but are more frequently identified by one of the following: certain steps a teacher should follow (Galas, 1999; Penick et. al, 1996), asking questions guided by a categorization or coding schematic (Shaunessy, 2000), the type of framework for questioning or sequencing that is expected, e.g. guided cooperative questioning (King, 1994), guided reciprocal peer-questioning (King, 1990), student questioning (Commeyras, 1995), questioning, listening and responding (Nicol, 1998-1999) and reflective teacher questioning. The research on coding or categorization of questions was not without controversy.

**Strategic Steps**

One series of strategic steps for questioning incorporated 5 steps that were utilized in an eighth grade classroom that incorporated technology in a student-centered
environment (Galas, 1995). The steps followed included: 1) engage students in asking questions. 2) individualize questions by asking students to describe their personal interests. 3) students categorized or grouped questions and/or eliminated questions with justification. 4) exploration and explanation of conjectures and resulting answers, and 5) refinement of questions. and 6) continued exploration. Throughout the learning experience the teacher involved demonstrated to students that Galas (1999) valued “all questions, not just the ‘right questions’ where students delved deeper and deeper into neuroscience” (p. 12). The last step of the 6 step process was very cyclical in nature, where the teacher also asked students guiding questions as they explored, experienced and explored. This process was very similar to the process addressed by Simon (1994) that consisted of the following stages: 1) exploration. 2) concept identification. and 3) application which triggered a new exploration stage. One constant theme highlighted by Simon (1994) was that the ideas and processes continued to evolve as “the cyclical aspect of [this] framework emphasizes new learning always involves the application and extension of previous knowledge” (p. 78).

Another form of a stepped or sequenced questioning strategy was utilized to provide teachers with a tool for thinking about questions and use in science inquiry-based learning environment. This particular questioning strategy modeled questioning behavior by the teacher, which allowed students to visually recognize the teacher’s logic and reasoning skills for resolving problems (Penick et. al, 1996). A mnemonic was devised to assist in remembering the order of the stages, “H R A S E, which stands for history, relationships, application, speculation and explanation” (p. 26). The phase of speculation could lead to tangents in classroom discourse which could result in varying degrees of
productive discussions (Penick et. al, 1996). Penick et. al. stated that "questioning a student and listening closely to the responses allowed us to assess what students think and why they have that particular idea" (p. 29).

Categorization of Questions

A variety of questioning strategies was recommended, and some researchers believed that focusing the types or categories of questions on the higher level of Bloom's taxonomy would result in students' engagement in a higher level of thinking (Hamblen, 1984; Marzano, 1993; Shaunessy, 2000). Marzano (1993) and Hamblen (1984) summarized Bloom's taxonomy or classification system as containing six levels of cognitive processing:

1. Knowledge
2. Comprehension
3. Application
4. Analysis
5. Synthesis
6. Evaluation (p. 155)

Shaunessy (2000) and King (1990) identified application, analysis, synthesis, and evaluation as the higher end of the taxonomy spectrum and stressed the importance of teachers and students to structure questions at these higher levels. Teachers must be aware of the intended processes they want their students to use when structuring their questions. Shaunessy (2000) stated that the most commonly recommended types of questioning "is the divergent-thinking questions that probes beyond the convergent, one-correct answer question to delve more deeply into an area" (p. 15).
Karen Hamblen (1984) believed that the taxonomy was created on the belief that “learning occurs in a hierarchical fashion, starting with the simplest of thinking process and proceeding in a step-by-step manner through succeeding, more complex process” (p. 42). However, while Marzano and Hamblen incorporated Bloom’s higher regarded categories into their literary writings, both commented on the confusion that still exists in its practical use by researchers and teachers alike. Hamblen (1984) stated that “a major problem with Bloom’s taxonomy that continues to be a source of educational confusion and semantic embarrassment is the separation of the affective from the cognitive and the use of cognition to mean essentially intellectual and logical thought processes rather than all knowledge modalities” (p. 43). Hamblen (1984) also stated that another disturbing distraction of the taxonomy is the finite and minor distinctions made amongst the subcategories within each of the six levels. This has resulted in difficulty on the part of educators to classify behavioral objectives in line with the major categories themselves (Sitko & Slemon, 1982). Marzano (1993) agreed with and identified a specific example where “when teachers were asked to determine whether a specific question was an example of an analysis question or an evaluation question, teachers disagreed more often than not” (p. 155). Hamblen (1984) espoused “teachers should not be overly concerned with the classification of specific questions, but rather focus on formulating a substantive discussion that conforms to the spirit, if not to the letter, of the taxonomic sequence” (p. 49).

Framework of Questioning Sequence

Various research studies have been conducted that utilize one of the following frameworks for questioning or the sequence of questioning: guided cooperative
questioning, reciprocal peer-questioning, questioning, listening and responding, student questioning, and reflective teacher questioning. Each of these frameworks proved to have a dramatic impact on the instruction and the students’ learning.

King (1994) indicated that the effectiveness of the guided questioning strategy was attributed to the format of the guiding questions. Findings from this study suggested a strong correlation between the “level of questioning and level of knowledge statements, with integration (the highest level of questioning)” (p. 363). However, both students and teachers involved in this study received several hours of training on how to teach skills of explanation and question – generation, in addition to specific lesson materials to present at the five lessons (King, 1994). Similarly, the study conducted by King in 1990 on guided reciprocal peer-questioning also required training on behalf of the students on several occasions. The result of the study indicated that, “students who used the reciprocal peer-questioning framework asked more critical thinking questions, elaborated more and demonstrated higher achievement than their discussion group counterparts” (p. 675). King stated that “it was the question stems which actually elicited the high level of questioning and responding” (p. 681) and “due to the generic nature of question stems, it is likely that the same set of stems could be applied to any topic in any content area to promote thinking and discussion about that topic” (p. 683).

The framework of questioning, listening and responding prescribed by Nicol in 1998-1999, utilized an approach for prospective teachers. The result of this study showed the effects of questioning on teachers’ behaviors and highlighted that the experiences that each teacher had was different based on the individual perspectives that the teachers held on questioning.
Commeyras (1995) explored that teachers could learn from questions their students asked, where it focused on the learning outcomes for teachers rather than for students. Questioned utilized in this study on text comprehension focused on two categories: 1) information-seeking questions and 2) comprehension monitoring questions. This particular approach also required student training through teacher modeling and guidance. Commeyras (1995) stated that “thinking provoked by students’ questions can help us as teachers to reach broader understandings of text” (p. 103). Additionally, listening to and letting students elaborate the meaning of their own questions can provide insight into teachers’ thinking of recognizing there are a variety of way to approach and interpret the intent of a student’s question (Commeyras, 1995). It must be noted that when students raise questions through classroom discourse, some of these questions may be distracting, disturbing and provide moments of discomfort. However, teachers need to consider that these moments, as uncomfortable as the experience, provide for an additional opportunity to learn from listening to students (Commeyras, 1995). Similarly, Shaunessy’s 2000 study highlighted the direct impact of a questioning process that involves the students questioning and the assumptions inherent to the teacher’s question, which resulted in an opportunity for the teacher to pose better questions that eliminated the teacher’s assumptions.

The last and least utilized framework of questioning was that of reflective questioning utilized by Van Zee and Minstrell in 1997. Van Zee and Minstrell referred to this framework as a reflective toss which incorporates three major themes for using questioning: 1) help students clarify their meanings, 2) provide an opportunity for students to explore a wide variety of views in a neutral and respectful manner, and 3)
help students monitor the discussion and their own thinking. The result of the interpretive case study involved complex changes to student's thinking. The reflective toss sequence is described by Van Zee and Minstrell (1997) as "a student statement, teacher question, and additional student statements" (p. 227). This particular question generated by the teacher was intended to give the students responsibility for thinking. Minstrell described his intent as being "to elicit what students think" (p. 229) which is opposition to traditional teacher questioning which is to evaluate what students know.

An episode that was scripted recorded a series of reflective tosses that allowed Minstrell to engage all of his students in the exploration of one student's alternative approach to solving a particular problem. The exchange of statements and questions had discovered an alternative approach by one student. The effects of this reflective exchange of a student's alternative method allowed for Minstrell to formulate questions in ways that shifted the accountability for evaluating answers from the teacher to the student, resulted in the public affirmation in a change of a student's thinking, as well as engaging other students in the construction of the meaning behind another students' thought processes. A similar study by Gall in 1970 also investigated the questions and responses which engaged other students in responding to questions and statements of other students.

Van Zee's study allowed for the identification and reaffirmation by Minstrell of his underlying and emergent goals and beliefs to his instructional and questioning practices (Van Zee & Minstrell, 1997). The student accountability for validation and the analysis of questions and answers was also demonstrated in studies completed by Galas (1995) and Lampert (1990).
**Methods for Analyzing Questions**

A large focus of the literature reviewed was centered on transcripts of conversations and/or utterances that detail both teacher questions, student responses and additional follow-up questions, which could have been made by either the teacher or other students in the class, along with transcripts of debriefing sessions with teachers. Not all studies incorporated responses of the students, but merely tracked the type and amount of questions asked.

One consistent method for analyzing questioning and/or response transcripts involves either a coding scheme or categorization of questions, i.e. Bloom's taxonomy (Commeyras, 1995, Lampert, 1990, King, 1990, 1994; Nicol, 1998-1999; Shaunessy, 2000). Roth's 1996 research categorized scripted questions which utilized a typology specific to the content area of engineering.

A less recognized approach than analyzing questioning was an interpretative framework of reflective questioning was also reviewed (Van Zee & Minstrell, 1997). Minstrell was a winner of the 1995 Presidential Award for Excellence in Science Teaching, where Van Zee believed that Minstrell’s ways of speaking (use of the reflective toss metaphors) were critical to his cognitive approach to cognitive approach to instruction. The study focused its attention on the kinds of concerns that shaped Minstrell’s questioning, rather than to code or count question categories. As Van Zee stated, “from a theoretical perspective mutually exclusive categories cannot be defined in contexts in which there are likely to be multiple purposes for any particular utterance” (p. 235). These concerns as earlier stated were the three themes incorporated into the reflective toss sequencing, as well as Minstrell’s beliefs, underlying and emergent goals.
for instruction and student learning. In support of Van Zee’s comments, Shaunessy (2000) stated that “reflective questions encourage students to consider their thinking processes and examine their strategies in a metacognitive fashion” (p. 16).

*Other Influences on Questioning*

While there are numerous influences on both the use and ability of teachers’ questioning, three influences which were identified in the literature review were: teachers’ beliefs or perspective, lack of or enhanced amount of training or professional development, and gender of students in the classroom responding to or asking questions.

*Training*

Many of the studies that were written and reviewed incorporated practices of various types of training prior to or part of the actual carried out questioning strategy. However, lack of training or the need for teacher training in best practices of questioning was a continued reoccurring theme. The results of several training sessions had an overwhelming impact on the behaviors of the teachers and students.

One particular instance resulted in a shift of a teacher’s emphasis on asking questions of students in an effort to guide students processes in the direction she wanted them to take, to an emphasis on asking questions with the intention of engaging others in the whole-class discussion, which resulted in the teacher learning more about what her student’s were thinking (Nicol, 1998-1999). Another effect on exposing teachers to new questioning strategies was having provided teachers with the opportunity for personal reflection, which in turn provided for an opportunity to review instructional practices (Nicol, 1998-1999). When teachers focused there efforts on attending to students thinking and reasoning, teachers were provided with opportunities to investigate and
explore mathematical ideas and assumptions themselves (Nicol, 1998-1999). Nicol stated “that curriculum and instruction instructors can influence prospective teachers’ thinking and beliefs toward being responsive to powerful pedagogical principles” (p.63).

Commeyras (1995) viewed “training in question-asking skills as an educational innovation” (p. 102). Sitko and Slemon (1982) found that teachers felt that through training and having been provided with opportunities to explore questioning strategies, they were more aware of the levels of questions they asked and of the levels of students’ answers. Hall and Myer (1977) suggested that educators who are involved in planning in-service programs which are designed to create significant change in teacher classroom behavior should “consider teacher perceptions carefully” (p. 167). Other shortcomings were noted by Mitchell (1994) with regard to the interpretation of the skills by teachers and the actual application of skills. Mitchell’s 1994 study was conducted for the purpose of examining the possibilities of developing teaching skills models and structuring of teacher’s beliefs as they relate to a particular teaching skill.

**Beliefs**

Just as students bring with them prior knowledge and past experiences to a classroom, so do teachers. In addition prior knowledge and experience, teachers also bring to their classroom and ways of instruction their individual beliefs or implicit theories. In one instance a teacher that was observed became more aware of the contradictions in her own actions and beliefs when she posed questions that were intended to elicit student’s thinking (Nicol, 1998-1999). Nicol (1998-1999) went on to state that “different prospective teachers’ issues of questioning, listening and responding
arise in which one is in the fore than the background at different points in the course” (p. 61).

In a study completed by Hall and Myers in 1977 it was noted that “change in teacher classroom behavior closely ties and is complicated by teacher perception of performance” (p. 167). Mitchell (1994) described past research that concluded that teachers “hold and actually use beliefs to shape their practice, and that at times there beliefs tend to be idiosyncratic” (p. 70). Mitchell (1994) concluded that the beliefs held by teachers are often incomplete and inconsistent with those from skill models in other teaching programs. However, this is in direct opposition of the study completed by Van Zee and Minstrell (1997). Since Minstrell is regarded as a teacher of excellence, his practices are not only guided by his beliefs, but also foster the emergent goals of instruction and student learning (Van Zee & Minstrell, 1997).

Data from Mitchell’s 1994 study indicated that the theories held by the individual teachers varied in degree of development, although each theory was based on an initial, stable set of beliefs about questioning skills and the purpose that they served. Mitchell (1994) viewed the elements of the beliefs to be non-contextual. Van Zee and Minstrell (1997) assert that the interpretative framework for reflective questioning “modify the way they envision interactions with their students in the classroom” (p. 259). Simon (1994) indicated that “the exploration of students’ mathematical thinking involves the application of teachers developing theories of mathematics learning” (p. 84).

Gender

In the study conducted by Roth in 1996, he observed and analyzed videotapes of the interactions of a teacher (Gitte) and the students from fourth/fifth grade inquiry
classroom. The analysis was completed using a sociolinguistic framework. The teacher noted that "every time she asks a question, the interactions with students are mediated by other aspects of the setting, such as gender of the student, whether the situation is a small, single-sex group, or whole-class discussion" (p. 709). Both the observer and the teacher noted that while there were no gender achievement issues, there were gender-related differences in the level of engagement in classroom discourse.

During an episode of conversation exchanged between a small group of 3 girls and 3 boys, the teacher had felt she achieved a balance of gender, when in fact the students had complained. The girls indicated that they did not want to be selected or frequented for participation and the boys complained about preferential treatment. It was observed that females participated completely when conversations were one-on-one with another female or with the teacher directly, but entirely different in whole-class discussions. This was mitigated by two other episodes that clearly demonstrated a distinct gender-related level of participation in whole-class discussion where the teacher herself, did not make an attempt to balance the participation of males and females within the class. Girls contributed little on their own and were almost never selected as the next speaker by either the teacher or the present speaker (Roth, 1995).

In a study completed by Taole in 1995, he observed through observations similar evidence which resulted in boys indicating their willingness to answer questions more than girls and that there was a significantly higher occurrence of the teacher calling on than girls. Unfortunately, also in this study there were records of "the teacher making disparaging remarks to the girls for either not doing their work or getting something wrong in class" (p. 268). In one particular lesson it was observed that the teacher was
more attentive to boys, “ignored the girls” (p. 269), and provided encouraging remarks to boys who answered questions and remarks with negative connotations to the girls (Taole, 1995). As a result of these observations, practices were established where teachers were asked to keep a class list and record every time they called on a student to answer a question in class. Prior to lessons, teachers were to identify specific females to whom they were going to direct questions and encouraged to choose females who were not normally active in class (Taole, 1995).

Summary

The review of literature and studies have suggested that various questioning strategies have a dramatic impact on teachers’ instruction and beliefs, students learning and thinking skills, interactions in the classroom. As Gall (1970) stated, “teacher’s questions are of little value unless they have an impact on student behavior”, (p. 714). Additionally the effectiveness of these strategies are also influenced by other factors such as the extent of training, beliefs and/or theories, knowledge of content and questioning strategies on behalf of both teacher and student, gender, social settings, and time for planning and reflection by both teacher and student.

The literature implied that continued research should be performed to develop models of appropriate framework for teachers to be exposed to. In order for teachers to develop their questioning skills and continue their modifications of knowledge and beliefs, continued professional development is required. In order for professional development opportunities to exist, teachers need the support of other professionals and administration that recognize not only the benefit of questioning strategies, but the need for them as well for both the teacher and the student. The support of administration
should come in the form of similar schedules for staff that teach the same or related content, time for planning and collaboration, and continued training opportunities. If content and context are expected to change, so must the ways in which teachers and students think about and experience them (NCTM, 2000). Teacher training should involve both the study of questioning strategies and guided practice in their use (Gall, 1970), as teachers would guide their students. Shaunessy proclaimed that “through the modeling of questioning and appropriate behaviors, educators and parents can encourage students to move into the role of facilitator, which is essential to the development of lifelong skills and growth as an independent learner who asks questions about texts, research and life” (p. 19).
Methodology

Teachers have an opportunity to learn many things when teachers themselves, as well as their instructional practices and interactions with students are investigated or researched. Past researchers have shown that teachers who investigate their own instructional practices: 1) tended to develop a more critical perspective on their own practices, 2) when teachers were learners in the mathematics setting, the knowledge gained by teachers stimulated important changes in the perspective, ideas, and understanding that many teachers had to mathematics, 3) provided for reflection by teachers of the learning environment, and 4) provided opportunities for teachers to focus on the teaching strategies they were utilizing and learning about (Simon & Schifter, 1991).

The approach utilized in this research combined the ideas of process-product research with that of sociolinguistic approach that highlighted aspects of teachers practice, focused on teacher’s questioning and the impact it had on student’s behavior.

Participants

Three regular education mathematics teachers from one suburban high school volunteered to have themselves and their respective Math B Regents (R) classrooms involved in the research on questioning strategies, which were intended to provide teachers with acknowledgement of existing practices or with the opportunity to develop their teaching practices of questioning designed to promote critical thinking skills on the part of their respective students. While one Consultant Teacher Services (CTS) special education teacher and one paraprofessional are acknowledged as being present in one classroom that was utilized in the study, other than their own questions or remarks being
included in the various interaction transcripts, they did not contribute to the analysis, evaluation or debriefing discussed. Both the CTS special educator and paraprofessional agreed to have their comments or utterances made within the confines of the classroom as part of this study.

**Teachers**

The three teachers hereafter referred to as Teacher 1, Teacher 2, and Teacher 3, each had varying years of experience in teaching. Teacher 1 had taught for a total of eight years, fours years in the late 1980's when pedagogical teaching practices were quite different, in addition to her most recent three and a half years of teaching at Penfield High School (PHS) and one semester at East Irondequoit High School. She had previously taught Math AR, Math AR Expanded, Math A/B R (two years), Geometry, Computer Science (CS) 1 and 2. At the time of the study she was teaching three sections of Math BR, one section of CS 2, and one section of AP CS.

Teacher 2 had three and a half years teaching experience, 2 of which were at a different district, but included prior experience with teaching Math BR. Teacher 2 had taught Math AR and Math BR and was currently teaching two sections of Algebra R and three sections of Math BR at the time of the study.

Teacher 3 came to the study with six years of experience, four of which were at PHS. Teacher 3 previously taught Math AR, Math A/BR, Math BR, Math B/IV H, Algebra/Trigonometry, Math IV, and Math IV H. At the time of the study, Teacher 3 had been teaching 2 sections of Math BR, one section of Algebra/Trigonometry and one section of Math B/IV H.
Students

Each of the classrooms involved in the study ranged from 25 – 30 students which contained a diverse population of primarily eleventh graders, both male and female. There were at least two accelerated tenth grade students in one class, in addition to one senior who was taking the course for the first time. Teacher 1 had selected two classes for participation in the study. One of Teacher 1’s classes of 25 students was itself already involved in modeling a new special education CTS support model, where a CTS educator and a paraprofessional, alternated every other day in providing “support” for six students that were classified as special education students who had Individualized Educational Plans or IEPs. Additionally, there were two other students who were classified as special education students who each had a 504 accommodation plan. The 504 plans had been established for individuals who have a physical or mental impairment which substantially limits one or more of their life activities as described in section 504 of the federal statute. While this particular class may not have been viewed as typical inclusion class due to the CTS support, Teacher 1 felt very strongly that her instructional strategies used in this class as well as the other, were actually more important than perhaps those that she utilized in her other classrooms in order to accommodate the special needs of the special education students. The other class selected by Teacher 1 provided her with other challenges, as it was a full class of 30 students.

Teacher 2 and Teacher 3 each had selected one of their Math BR classrooms for the participation in the study. Teacher 2 selected her particular classroom for consideration based on the students historically requiring a great deal more direction and assistance on her part than her other two Math BR classrooms. Teacher 3 selected this
particular section of Math BR as she wanted to enhance the learning environment of this class to provide for a more student-centered classroom, where she could gain more knowledge of her students' ability and understanding prior the student performance on a formal assessment. Teacher 3 thought the opportunity to review her current instructional strategies and questioning techniques, would enable her to make the necessary enhancements or modifications that she hoped would result in more student engagement and communication.

Instruments and Materials

The instructional materials consisted of a one lesson on the imaginary numbers and the cycle of $i$ which served as a foundation lesson for the teachers to analyze and structure their questioning techniques (see Appendix A), and a series of lessons on a Law of Sines and Cosines consistent with the New York State Math BR curriculum. The lessons on the Law of Sines and Cosines utilized an investigative approach from the class text (see Appendices C – E and Appendix J (Coxford, A. R., Fey, J.T., Hirsch, C.R., Schoen, H.L, Burrill, G. Hart, E.W., Watkins, A.E., Messenger, M., Ritschma, B.E., and Walker, R.K., 2003). The teachers all agreed a return to a more inquiry-based lesson style was more conducive to questioning strategies, both formal and informal.

All of the activities were carried out in the students’ classroom environment with their regular teachers, where the interaction between the teachers and students was recorded by the teachers and the CTS and paraprofessional staff members previously identified in the study. During various parts of the lessons, students were paired together to work on various problems designed by the teachers. At the end of at least two lessons, informal assessments were provided to the students as ticket-out-the-door where students
did not need to put their name on the document. This information was reviewed and utilized by the teachers to modify or enhance the following lesson, homework assignments and anticipated assessments.

For the development of the second lesson, teachers were provided with a portable flip chart of the six main categories of Bloom’s Taxonomy for review and consideration of, in preparation of the questions that would be utilized in the subsequent lesson (Barton, 1997). The teachers developed questions for instructional material for each lesson and two assessments on the Law of Sines and Law of Cosines based on the feedback obtained during the lessons, discussions from homework and in-class problems, tickets-out-the-door, and post-lesson discussions.

Equipment utilized by the teachers and students included TI-83 graphing calculators, overhead projector with teacher-overhead TI-83, and a desktop-pc connected to ceiling mounted projector for instructional displays. Other materials included student handouts, student copy of classroom text Core-Plus Mathematics Project, Contemporary Mathematics in Context, and rulers.

Data Collection

Data sources for the discussions and analysis conducted by the teachers included transcripts of interactions of the respective teachers and their students (see Appendix B), notes from conversations during the debriefing sessions, samples of students work obtained from homework, warm-ups (activity at the beginning of class meant to tie in to either prior or current lesson material) in-class partnered activities, “tickets-out-the-door”, and a sample of student responses to assessment questions (see Appendices E-H.
and M). The identity of each student was anonymous where the question and/or response of the teacher and student were indicated as such.

The debriefing sessions included discussions that reflected on the lessons the same day that they were conducted, notes from the analysis of and discussion of the recorded transcripts of the foundation lesson, impact of questioning on the students' learning and behavior, the impact of the responses from questions on the teachers' behavior and modification to instruction on future lessons and assessments. The analysis and discussion of the transcripts for each lesson were conducted during a series of meetings after school during the following weeks and numerous subsequent discussions. The process followed was similar to that followed by Nicol in 1998-1999, Van Zee and Minstrell in 1997, and Simon in 1994.

Procedures

The teachers met periodically to develop four different lesson plans geared towards the analysis of questioning utilized in the classroom by both the teacher and the students. Each of the four lessons included the content to be discussed, essential questions to be covered, the learning environment, homework (if applicable), and the connection of the lesson to prior and future content.

The first lesson was developed to establish a baseline of the questioning strategies that each teacher utilized as their inherent instructional practice, in addition to the types of questions and responses that the teacher and students made during the various stages of classroom instruction and settings of learning environment. The first lesson extended the students' process of expressing the square root of a negative value, specifically solutions
to quadratic equations that resulted in the square root of a negative value, to be expressed differently in terms of \( i \).

The teachers agreed on the main ideas and concepts that needed to be covered in the lesson, as well as the format and content of the instructional material that was utilized (see Appendix A). No discussion took place with regards to what questions should be asked or expected difficulties that students could experience. The teachers met after school once the lesson was carried out and discussed their respective initial reflection as to what had transpired in their classrooms. Once the recorded transcript was provided to the teachers to analyze, they met to place the various questions into categories that they felt reflected the nature of the interactions of the teachers and students.

The teachers were then provided with a portable index-card sized flip chart that contained the six main categories of questions as identified in Bloom’s Taxonomy (Barton, 1997; Gall, 1970, et. al.) to discuss the how the questions previously recorded and analyzed into categories compared to those of Bloom’s Taxonomy. Teachers then discussed whether or not the newly identified categories would affect the type, occurrence of and development of questions that they hoped would foster student critical thinking for the remaining three lessons to be studied.

The teachers worked collaboratively and developed lesson plans that incorporated questions the teachers planned to ask that would guide instruction, as well as guide students thinking with the possibility of having students ask the important content questions. The teachers also identified potential pitfalls in misunderstanding or processes that could be expected in the students understanding and performance of their skill with regards to the Law of Sines and the Law of Cosines. The questions developed by the
teachers incorporated the objectives established by the authors of the classroom text and levels of Bloom’s Taxonomy (Barton, 1997; Gall, 1970, et. al.). The process of debriefing and analysis continued over the course of the development of and completion of the remaining three lessons which covered Law of Sines, Law of Cosines and a mixed review of both concepts. A record of any subsequent discussion was also made.

Upon the completion of the four lessons, the teachers met to identify and discuss the overall results and the impact the research process had on their instruction, behavior, understanding of content, behavior of students, students performance with problem-solving on informal and formal assessments, students critical thinking skills, and any other observations. Success of the study was measured in a more qualitative analysis through exchanges of dialogues with amongst the teachers, teachers with students and students with other students. Teachers agreed that if the research process has a either a positive impact on the teachers' instruction in the classroom and or students performance determined in a subjective manner, the study was successful at some level. From a quantitative perspective, teachers analyzed the effect the lesson study process and review of questioning strategies had on students' performance by the analysis of grade improvement on the last assessment from prior student performance. Each teacher looked at individual student improvement, as well as overall improvement for the respective classes. To the extent students and/ or teachers experienced improved communication, both in frequency and quality, verbal and/or written, effected teacher and/or student behavior with regards to mathematical thought process, critical thinking, knowledge of content and problem solving ability, success was achieved.
Results

The teachers revised the methodology slightly as a result of the initial debriefing that followed the conclusion of the first lesson. The teachers had originally planned on categorizing the questions contained in the questioning scripts from the first lesson into an agreed upon categorization scheme and to have compared the types of questions with those of Bloom’s taxonomy. The teachers wanted to identify if they had asked questions that would have been viewed as critical thinking questions or those that would have led students to the development of critical thinking.

The teachers did not feel that the lesson on the Introduction to Imaginary and Complex Numbers (see Appendix A) was very thought provoking, although it was on an abstract and new idea. The teachers made several reflective comments regarding the lesson transcripts (see Appendix B). Teacher 3 observed that her students were comfortable in expanding out factors using a -1 as an additional factor and very willing to share work on the board. Teacher 1 observed that the students in her fourth period class recognized the connections to the rules for exponents and some students recognized and were able to explain the importance of i as part of the coefficient in isolating the radical to other students in the class. Teacher 1 commented that the students in her 9th period class asked great connection questions. She provided samples of the students’ questions that were asked which were content specific, such as did it matter where i was in the expression? She noted that another student in the class answered the student’s question instead of herself. Another student asked if \( i^2 \) was possible and would it be handled like x when combining like terms. Teacher 1 also stated that she did not get to the patterning
cycle of $i$ and that some of the students seemed to have struggled with a connection to the
rules for the exponents, although students may not have had enough time.

Teacher 2 commented that she took too much time with the warm-up and the
homework and did not have time to cover the original lesson. She felt the questions
asked made the students think. She restated the student's questions for the whole group
to respond to rather than the teacher. Her students did not recognize the values of the
earlier patterns of $i$ as other powers of $i$ and she was concerned about the success or level
of difficulty the students would experience with the homework assignment. Teacher 1
and Teacher 2 both indicated that it was extremely difficult to record and track both the
teachers' and students' questions and responses accurately.

While the review of the lesson provided the teachers with an opportunity to see
what information the students had remembered about some of the rules of operations with
exponents and allowed for students to expand their knowledge about how to express roots
for a quadratic equation in another form, the teachers all felt that different types of
questions needed to be utilized in the classroom other than those that determined what
information students recalled. They decided rather than categorize the questions into
process, connections, etc., their time was better spent on the development of more
meaningful questions that were in line with learning objectives that were focused on how
the students thought when solving problems.

As a result of the collaborative lesson planning sessions, three lessons that
covered the Law of Sines and Cosines, one lesson that introduced each topic and a third
that provided for mixed review (see Appendices D, I, K, and L) were developed.
Question 5 from Appendix D was omitted from the lesson initially and was used as part
of a homework assignment later. The checkpoint at the end of the investigation was used the next day after students had an opportunity for application practice and was given to the students separately (see Appendix F). The teachers decided that the development of the questions and their success in the classroom with regards to the use of questioning strategies was impacted by the instructional materials and environment used for these lessons. The teachers went back to an inquiry-based approach to instruction, which meant going back to the classroom text, Core-Plus Mathematics Project, Contemporary Mathematics in Context. However, the teachers broadened the scope of the instructional material which included guiding questions developed in line with sample questions from a tool that Teacher 1 remembered from her new teacher training four years earlier called the Quick Flip Questions for Critical Thinking. The questions were from:

**Level III Application:**

How would you use...?

How would you solve_____using what you’ve learned....?

What other way would you plan to....?

**Level IV Analysis:**

How is _____related to ....?

Why do you think.....?

What is the relationship between....?

Can you identify the different parts...?

**Level V Synthesis:**

What would happen if...?

Can you propose an alternative....?
Suppose you could... what would you do...?

Can you formulate a theory of...?

Level VI Evaluation:

What is your opinion of...?

Why did they (the character) choose...?

Would it better if...?

How could you determine?

How would you evaluate...?

Based on what you know, how would you explain...? (Barton, 1997)

An additional result of the collaboration process was that the teachers decided it was necessary to utilize a preliminary investigative style lesson prior to the lessons developed for Law of Sines and Cosines that were evaluated for the impact of questioning and questioning strategies. Students had not used their text in some time and the teachers felt it was necessary to have the students become reacquainted with the text and the style of the lesson. The lesson utilized by the teachers immediately preceded the Law of Sines investigation in the text (see Appendix C). Informal observations from this lesson were made in order to make a more comprehensive lesson on the Law of Sines which incorporated pitfalls for students’ miscalculations, as well as prior experiences with related material. Some students were weak on basic right triangle trigonometry that impacted their success early Law of Sines, as well as their understanding to the development of the area of a triangle formula which came later in the material. The investigation and the “On Your Own” (OYO) page 27 in Appendix C, as well as warm up
problems served as an informal assessment for the application of basic right triangle trigonometric function of Sine: Opposite over Hypotenuse, Cosine: Adjacent over Hypotenuse, Tangent: Opposite over Adjacent (SOHCAHTOA listed on student work in Appendices G and M).

The teachers reviewed extensively the investigation on Law of Sines (see Appendix D) and developed questions that were included in the lesson, as well as an alternative way for the introduction of material that they felt started the students' process of thinking by having them make connections to what they knew or they thought they knew. In anticipation of some of the expected responses, the teachers identified possible questions that would guide students thinking or redirect it as needed. The questions developed were:

1) Will students want to use the Pythagorean Theorem?
2) Will they identify the distance in the picture from the fire to the tower as a hypotenuse?
3) Can you find the missing side?
4) Why can't you use Pythagorean Theorem with the new picture?
5) Can you use right triangle trigonometry?
6) How would you use right triangle trigonometry to find the values needed?
7) Could you check your answers with the Pythagorean Theorem?
8) After introducing the Law of Sines: Will you use all three fractions to solve for the missing values? Why are Why not?
9) Which ratio do you omit?
10) How do you know which one to omit or which ones to use?
Teacher 3 cautioned the other teachers that students in the past had made particular errors with the Law of Sines relating to which side lengths and where the students place the side lengths in the ratios. Her past experience indicated that students brought in the adjacent side to the angle rather than the opposite side, primarily because the students had labeled the picture wrong. Teacher 1 commented that it would be a good idea to ask students where the largest side is in relation to the angles in the triangle, etc, which could eliminate this error for some students and it would also provide for the opposite relationship that is modeled in the law of sines. This type of error did occur as part of the Law of Sines practice performed by the students, but the teachers had the students explain what was done incorrectly rather than the teachers themselves.

In the Law of Sines investigation (Appendix D) students were asked to analyze the problem of a non-right triangle but it was drawn on the board rather than provided to the students initially. Students were asked to brainstorm ways to solve the problem. In each of the classes, each teacher had at least one student who suggested using the Pythagorean Theorem. There was at least one student in each class that explained why this was not an option initially, but would be only after the altitude was drawn in the picture. Only Teacher 1 had one student who realized the law of sines was going to be the new alternative, but could not remember the "formula". When the student saw the formula on the next page, Teacher 1 asked the student how many and which ratios she would need to use. The student replied, "I'm not sure, I'll have to figure it out again". Each teacher than asked the student who proposed the solution to provide the first set up of an equation that would be needed to solve the problem. The classes were than told to finish the problem.
Many of the students asked to have his/her answer validated as they worked with their partner. Once the teachers had the students share out their responses for the answer, the teachers each had the students decide which answer was right. The students determined who was right initially by either how many people had the same answer or if it matched their own. Each teacher had a student explain the steps taken that led them to the right answer. Some students who had not obtained the same solution found their own mistakes, while some other weren’t sure what they had done wrong. The next question raised, in some instances by the teacher and in others by the student, was “Is there an easier way, one with fewer steps?” From one of Teacher 1’s students, “Is there a shorter way so I won’t make so many mistakes?”

The teachers then directed the students to complete the next few questions in the investigations, where the teachers monitored the students work and discussion, providing individual explanations as needed. Some students had difficulty with questions 2 and 3. Question number 2 made the students interpret an existing solution to the problem, so some students did not understand what the text was asking as question 2 as the problem already had the steps listed to the solution of the problem. Question number 3 had no values at all and had the students apply the process used in the prior problem to a more generalized, abstract portrayal of the same process. Teachers regrouped the students after the completion of number 3 to solidify and restate the developed equation of the Law of Sines. This was when the teachers incorporated some of the questions above as to how many ratios were needed and how the students knew which ones to use. Positive responses were made by several of the students, although there were several students who did not elicit a response. The homework at the end of the investigation (see Appendix E)
and the checkpoint that followed (see Appendix F) provided the teachers with each
individual’s level of understanding at of the Law of Sines at that point in time.

One unexpected result of the OYO on page 31 (see Appendix E) was that students
solved for both missing lengths, rather than utilizing the principle that the smaller angle
would be opposite the smaller side and did more work than was needed to determine
which airport was closer. Other students did not interpret their own solution correctly,
because they did not put the results of the equations back into the picture or put it back
into the diagram incorrectly. Several students rounded very early in their calculations
which if asked for the difference of the distances, the students would have gotten the
wrong answer. In the past, students have lost points due to rounding errors, either
completed too soon or by not using enough significant digits. The teachers addressed the
rounding issue throughout the lessons. Teacher 1 had another student who had
interpreted the language of the problem very literally and solved for the correct distance
because it said the “nearby airport” (Coxford, et. al. 2003) and so he did not bother to
solve for the other side.

The checkpoint (see Appendix E) was utilized by each teacher in varying forms.
Teacher 3 used the checkpoint as questions she used as part of a facilitated class
discussion. Teacher 1 drew the problem on the board for her period 4 class and
facilitated a class discussion, and obtained solicited responses from students. She did this
for her period 4 class as they had demonstrated difficulty with a similar checkpoint that
asked abstract questions in the lesson that reviewed right triangle trigonometry (see
Appendix C). She decided to expose the students to the same abstract questions, but did
not have them struggle with the questions as they were originally phrased. She
paraphrased the questions and broke them down looking from a particular perspective in the triangle. This was similar to what she had done previously with right triangle trigonometry. She had used the map in the mall idea, where she drew a stick figure and said, “Solve when you are here →”. However, Teacher 1 (ninth period) and Teacher 2 assigned the checkpoint for homework and used it to launch the discussion for their next class. The results from the discussion showed students either had a very strong sense of the manipulation needed for Law of Sines by the representation of multiple and accurate equations (see Appendix F) or were did not understand how to represent the formulas when no numbers or contextual setting was provided. Some students provided incomplete written, non-equation explanations or incomplete explanations. Student 2 in Appendix F provided written and equation formatted explanations from multiple perspectives in the triangle. This particular student provided an accurate representation of solving for the missing angle using the inverse sine function, which has typically been an area that student have difficulty retaining. Some students could provide equations when looking for a side and show the equation in terms of the desired side, but could not accurately write the same type of equation when looking for the angle.

Other results obtained from student work (see Appendix G and H) showed some students have difficulty formulating a picture to represent the situation if it was not already provided. This also occurred when only angle and side measurements were provided without a contextual application. In some instances, students correctly drew and labeled the picture, solved the problem correctly, but misinterpreted the results. This result occurred on the Quiz in Appendix H, although the two samples included here had correct answers. In the problem shown in Appendix H, some students had labeled the
sides and angles of the triangle as a, b, c and A, B, and C respectively, however the
defenders to the quarterbacks were also labeled defender A and B, which resulted in
some students solving for side a, but then referring to Defender A, which was at the end
of side b. Some students still had difficulty when the sine of an angle was provided as a
decimal instead of having either a fraction provided or just the angle itself (Appendix H,
extra credit section answer not provided).

Another change in methodology occurred when the teachers had recognized the
mixed results when the students were asked to answer abstract, generalized questions
about when and how they would use the Law of Sines. The instructional material
selected, specifically for original lesson for the Law of Cosines (see Appendix I), was
revised (see Appendix J) as a result of the continued collaboration and debriefing
sessions, based on observations made by the teachers and the perceived needs of the
students. The teachers did not feel that the textbook lesson was developed with the same
intention as the Law of Sine Investigation. The lesson did not have the students discover
the Law of Cosines, nor did it provide for the application of the law with any numerical
examples until the end of the investigation. The teachers did utilize the more abstract
application of the Law of Cosines at the end of the investigation similar to the checkpoint
in Appendix F. The teachers utilized a resource that other Math B teachers had
developed and utilized previously. However, the teachers discussed at length the strategy
that was to be used and the questions that they had hoped would result in their students’
deeper understanding of how and when the Law of Cosines should be used and how the
formula changed with different situations. Teacher 3 provided guidance into expected
pitfalls that students experienced in the past, which was invaluable. Teachers helped
students recognize these mistakes and reviewed necessary steps such as PEMDAS (parentheses, exponents, multiplication/division, and addition/subtraction) and the necessary information that was needed to have the Law of Cosines utilized as a problem solving strategy. Once students had been exposed to this application, the teachers asked students why they could not use the Law of Sines for solving the problem and vice versa. The teachers also asked students why the Pythagorean Theorem could not be used with these types of problems. If students were uncomfortable with applying either Law of Sines or Cosines, they reverted back to the Pythagorean Theorem, even when no right triangles were provided.

The last lesson consisted of 4 application problems with no pictures (see Appendix K). The teachers allowed students to start this in class where assistance was provided if needed. Several students struggled with the geometric reference to a parallelogram. Many students did draw the picture, but they solved for the wrong diagonal and did not realize it. The teachers decided that additional review was needed that consisted of a more diverse mix of problems that would result in better accessibility for students, which included some problems with pictures, some without and some basic right triangle trigonometric applications (see Appendix L). Those students who struggled with the basic trigonometry had another alternative to the same problems if they used the Law of Sines, which many did use.

The use of questions developed by the teachers resulted in increased communication between the teacher and students, teacher with other teachers, as well as between students. Teacher 1 observed that several of her students asked each other questions and provided explanations to each other about why they did what they did, or
how come etc. The same students watched Teacher 1 to make sure that it was okay for
them to have having their conversations and that they weren’t missing out on anything that
the teacher was saying. Teacher 3 remarked that a student came up to her and said, “I
know I can’t get credit for my answer, but I know I did something wrong and I want to
figure out what it was.” Teacher 2 thought her students had not increased their
communication with her, but she had observed an increase in student to student
communication similar to that of Teacher 1.

The teachers were surprised as to the success the students had with the
manipulation of the Law of Cosines and to the comfort level the students showed with
knowing which version of the law to use. The teachers decided to ask their students
which law they were more comfortable with, but more importantly why. Teacher 3 had a
student who commented that she thought the Law of Cosines was easier because the
student just recognizes the equation and she knows where to plug everything in and solve
it. The same student explained that when she had to use the Law of Sines, it was
confusing to her because the opposites and angles always seem to be move around.
Teacher 1 had a student who commented that the Law of Sines was easier because she
always knew that she would have to have something across from something else, where
only one thing is missing out of the two sets. The student explained that it was easy for
her to look for the related opposites. However, the same student stated that she still had
trouble when she had to solve for the angle when she thought she couldn’t use Law of
Sines. The student said when she looked at the picture it always appeared that she was
missing information that was need for her to substitute in the equation. Teacher 1 made a
special note of this last comment, as she knew it would resurface when students would be asked to solve for an area of a non-right triangle, where no angles were provided.

The communication initiated by the teachers and the candid responses made by the students made the teachers investigate and incorporate these concerns and inabilities in problem solving in future problems in other topics not yet covered like area of a triangle, resultant force and other mixed applications.

The questioning strategies also had an impact on students' behavior with regards to problem solving strategies. When the teachers graded the assessment on Law of Sines, Cosines and Basic Right Triangle Trigonometry (see Appendix M), not only was there a marked improvement in test scores for many of the students, students demonstrated an increase understanding of the geometric vocabulary, as well as the demonstration for alternatives to solving problems. Teacher 3 had two students that demonstrated strong algebraic processes and understanding of the Law of Sines, but solved for the wrong side, correctly. Teacher 1 had a student apply the Law of Cosines to a problem who did not follow the rules of PEMDAS and obtained an answer of 0. The student recognized that this could not be possible and utilized the properties of an isosceles triangle to find enough information to correctly solve the same problem correctly using the Law of Sines. In the past the same student would have crossed out the bad work and would have given up. Teacher 1 and Teacher 3 each had students that got stuck when using the Law of Cosines because they forgot to substitute the angle in the formula. Since it was the isosceles triangle, they knew what they needed and switched to the Law of Sines and solved the problem correctly. Teacher 1 and Teacher 2 each had two students that had recurring difficulty in remembering when and how to use the inverse sine function. As
an alternative on the test, the students utilized the Pythagorean Theorem appropriately on a right triangle and then utilized the Law of Sines as an alternative to SOHCAHTOA.

The overall research and collaborative process impacted the teachers instruction in not only the preparation of the material but how and what the teachers thought about the material, as well as how and what the teachers thought about how and what the students learned. Teacher 1 commented that the collaborative process enabled her to make stronger and deeper connections across the topics within the curriculum. Not only did she have a better sense of what her students were learning, but she also had a better sense of where she needed to take their learning and how they were doing along the way. Teacher 1 felt that she was better prepared to handle her students’ difficulties herself, but more importantly to enable other students to help as well. Teacher 3 stated that the collaborative process enabled her to develop more meaningful instruction and explanations on topics that she had taught before to accelerated students, but found better ways to disseminate and connect the information for Regents level students. She went on to say that she was allowed an opportunity to make new connections, relationships and new ways to teach. Teacher 2 commented that she was able to make her classroom more student-centered, where she became the moderator while the students discussed the topics and process. She had other students answering the questions of other students. Teacher 2 realized that she had become a better teacher when she had a better understanding of how everything was connected to what she taught. She felt that she was better prepared to handle a larger variety of questions by her students. Teacher 2 also was able to provide responses that were scaffolded which resulted in a variety of answers for what her students needs were. Teacher 2 commented that by experiencing the questioning process
she now understands why questioning is so important. Teacher 3 commented that during
the process of thinking of possible questions, she focused more on the objectives of what
the students should be learning. The questioning helped her realign her teaching, made
the instruction smoother, tighter, and brought relevance of the detail together. Teacher 3
realized the connections the students make to the content which allowed her to work on
deepening their understanding. Teacher 3 also observed that there was a shift in the
attitude in learning the math from just looking at grades. Teacher 3 said, “The kids
wanted the quizzes back because they told me they knew that they had done well, and
they had.” Teacher 1 noted that the kids were thinking about the strategies to solve the
problem, as well as their solution and whether or not it made sense. Teacher 3 concurred
that she had seem the same thing. She said, “The change in instruction has impacted what
and how the kids think.” She went on to comment that she was very glad that the
questioning process also had impacted her mindset as well. Teacher 3 felt much more
positive about what and how she was teaching her students, as well as how they are
learning. She had commented at the beginning of the process that going back to an
inquiry-based approach was definitely needed because she felt like she was losing my
repore with my students. As a result of the collaborative process to develop questions and
instructional material, the environment in Teacher 3’s classroom became much more
engaging, student-centered and positive for everyone involved.

All three teachers agreed that they were having more conversations about how the
students were learning, not just what they were learning. The teachers asked questions
about why the students did not understand the material with a desire to fix or uncover
what wasn’t working. In the unit prior to addressing questioning strategies, Teacher 1
shared that when she had made statements about the students not getting it before, the comments were much more negative. Now she wanted to really talk about why the students were getting the material with her colleagues to see if the same thing was happening in their classes and if and how they were addressing the problems. Many of the results are subjective, but the results of this research were more qualitative then quantitative.
Discussion

The teachers experienced first hand many of the issues discussed in the literature reviewed. The teachers were first faced with whether or not it was important to categorize the questions that were recorded in the transcript of the first lesson. The teachers decided to incorporate the levels of Bloom's Taxonomy as suggested by Shaunessy (2000) and King (1990), but concurred with Hamblen (1981) that the categorization of the questions would provide for a distraction. Instead the teachers focused more on the formulation of the questions and responses by the students that conformed to the spirit of the taxonomic sequence (Hamblen). The teachers wanted to have lessons that were based on a guided questioning strategy which they had hoped would result in effective instruction similar to that of King (1994). Unlike King's 1994, the teachers had not participated in any training, other than what they had each been exposed to in their graduate programs and prior professional development. However, like the question stems utilized by King, the teachers utilized similar stems identified in Barton's Quick Flip Questions for Critical Thinking. The stems used in Barton's chart were similar to that used in King's study, as they were generic and could be applied to any topic in any content area.

The results of this study largely mirror components of Nicol's study (1998-1999), Commeyras (1995) and Van Zee and Minstrell (1997). The results of the research identified many things that the teachers learned from the questions the students asked, where the results included many impacts on the learning outcomes for the teachers, the behaviors of the teachers in how they developed their instructional materials, and how thinking provoked by students questions helped them as teachers to broaden their
understanding of the content. The teachers incorporated reflective questioning not only with students but with each other similar to that utilized by Van Zee and Minstrell in 1997. The teachers were more aware of not only the content but also more aware of how and what the students would experience through the problem solving process. Similar to the comments by Penick, et al in 1996, questioning a student and listening closely to the responses allowed the teachers to be more aware of what the students were thinking and more importantly, allowed the teachers to find out more about why the students had that particular idea, which in turn had an impact on the teacher’s instruction.

While the result of Van Zee and Minstrell’s study had an impact on the student’s thinking, the results of the research conducted by the teachers showed an impact on their thinking with regards to the content, their students learning, and their own instructional behaviors. Teacher 3 stated, “That I feel like I have reaffirmed my ways of teaching to make it meaningful for my students and myself.” This comment is paralleled to the comments made in Van Zee’s study where the reflective response process allowed for the identification and reaffirmation by Minstrell of his underlying and emergent goals and beliefs to his instructional and questioning practices (Van Zee & Minstrell, 1997).

The teachers in this study felt similar to that of Minstrell, where the students in both studies demonstrated their responsibility to think, rather than just what they knew. Teacher 1 and Teacher 3 both stated that the students were more accountable not only for knowing how to solve a problem, but that they validated the answer as well as the process in line with comments made in Galas (1995) and Lampert (1990). The results of this study reflected in statements by the teachers demonstrated that like Shaunessy (2000),
“reflective questions encourage students to consider their thinking processes and examine their strategies in a metacognitive fashion” (p.16).

The questions that the teachers incorporated into their lessons and assessments impacted the learning environment which made it more student-centered or engaged. Similar to Gall in 1970, the investigations conducted in the teachers’ lesson plans resulted in other students responding to questions and statements made by other students.

As with Nicol’s study in 1998-1999, the effect of exposing teachers to new questioning strategies provided teachers with numerous opportunities for personal reflection. Both Teacher 2 and Teacher 1 agreed that due to the discussion of expected mistakes and/or pitfalls that students could make and that have occurred in the past, made them not only better prepared to handle students questions and errors, but to be able to redirect these questions to the students to address and explain from their perspective instead of the teachers. This allowed the students to think about other students results and thought processes. This was supported by Sitko and Slenom (1982), although the study in 1982 had provided training for teachers. Sitko and Slenom said that when teachers were given the opportunities to explore questioning strategies, they were more aware of the levels of questions they asked and of the levels of the student’s answers. Teacher 1 and Teacher 3 commented on the significant impact it had on their mindset and perspective of not only the content, but their view of their effectiveness of teaching when they spent time on the development of the questions and the impact it had on their teaching, as well as the students’ learning. However, Teacher 2 was more tentative as to the overall impact the questioning process had her and her students. While Teacher 2 recognized a new importance questioning has in the classroom, she referred to the impact
it had on her more than the students. As Hall and Myers in 1977 indicated, "change in teacher classroom behavior closely ties and is complicated by teacher perception of performance" (p. 167). The slight difference in comments made by the teachers were supported by data in Minstrell’s’ 1994 study that indicated theories held by individual teachers varied in degree of development, although each theory was based on an initial, stable set of beliefs about questioning skills and the purpose they served. Like Van Zee and Minstrell (1997) the research conducted by the teachers here served as a beginning framework to modify the way they viewed their interactions with their students. The use of questioning strategies brought stronger meaning to the purpose of questioning for Teacher 2 and the reaffirmation of inquiry-based instruction for Teacher 1 and Teacher 2.

All three teachers recognized how questioning strategies allowed them to experience the exploration of students’ mathematical thinking and like Simon in 1994, recognized that such exploration involves the application of the teachers developing the theories of mathematics. All three teachers developed a stronger understanding of the content taught, as well as connections to future content the students would experience. The lesson development, implementation, debriefing meetings and continued refinement and reflection process that the teachers went through in this study were very similar to the stages identified in Simon’s 1994 study. As the teachers in his study went through the stages of exploration, concept identification and application, a new exploration stage was triggered. The cyclical aspect of both studies gave more meaning to the adage monitor and adjust that is usually a daily doctrine of most educators. As Simon (1994) stated, "the cyclical aspect of [this] framework emphasizes new learning always involves the application and extension of previous knowledge" (p. 78), but the teachers in this study
felt that this was not only true for their students but themselves as well. As Simon (1994) stated, “experiences with students’ learning of particular mathematics content contribute to teachers’ understanding of mathematics in general” (p.84).
Conclusion

The questioning strategies utilized by the teachers in this study had an embedded purpose within a particular content and context at times that contained components from Galas (1999) and Penick et al (1996) where the teachers followed certain steps that they developed, from King (1990), Barton (1997) and Coxford et al (2003) where they utilized a type of framework for questioning and sequencing developed with Bloom's Taxonomy and question stems in line with content objectives, from Commeyras (1995) and Nicol (1998-1999) involving student questioning, questioning, listening and responding and reflective teacher questioning.

The research conducted in the study demonstrated the standards by NCTM (2000) where "the teachers played an important role in helping enable the development of reflective habits of mind by asking questions and by anticipating the student's questions" (p.52). Additionally the teachers listened to each other and their students which allowed for all involved to develop and deepen their own understandings. The NCTM (2000) also stated that "conversations in which mathematical ideas are explored from multiple perspectives sharpen their thinking and make connections" (p.59). This statement was supported by the students as well as the teachers. The teachers in this study demonstrated that they not only learned from each of the other teachers, but more importantly from the experiences that they shared with their students by experiencing the content with the students both in and out of the classroom. The teachers in this study enhanced their knowledge coincident to Simon's (1994) "six key components of learning: knowledge of mathematics, knowledge about mathematics, useful and personally meaningful theories of mathematics learning, knowledge of student development of particular mathematical
ideas, the ability to plan instruction that adheres to the reformed principles and standards, and the ability to interact effectively with students” (p.72).

The teachers in this study experienced first hand a statement made by Penick et al. (1996) that questions raise new ideas and suggestions, which stimulate student thought and action while revealing a particular strand of problem-solving logic. However, to what extent the questions, questioning technique and questioning sequences utilized by the teachers had on the development of critical thinking and critical thinking was not determined. The time period and nature of the mathematical topics covered were too short and limiting for the teachers involved to have determined the overall impact they had on changing the status or level of the students’ critical thinking skills. It was clear that for many students, there problem solving ability had improved and both the students and teachers were thinking more about what and how they were learning. Teachers became more aware of the type of questions that address higher level thinking and that the need for reciprocal communication is major component to the success and development of student and teacher understanding.

Continued research needs to be conducted to develop training models and framework for teachers to experience and recognize the benefits for both the student and teacher. There is a need to help teachers develop opportunities for and provide them with the skills that enable them to provide for student centered questioning which requires teacher-student dynamics different that what is typically observed in classrooms. Continued professional development opportunities should be made available for teachers that will allow the teachers to develop their questioning skills, knowledge and educational beliefs to foster the teacher-student dynamics established by NCTM (2000)
which would provide for opportunities for teachers to develop their skill in determining what types of questions should students be encouraged to ask (Commeyras, 1995).

Another major component for teachers' success in the development and use of questioning strategies is the support of administration by providing for similar schedules for teachers who teach the same content. A major obstacle for the teachers in this study was the constant battle to schedule time for discussion, instructional planning, group reflection and modifications to instruction, as no pair of teachers in the study had the same planning period, in spite of the three teachers teaching at least two sections of the same course, in addition to one or more additional courses that they needed to prepare for. Most of the collaboration on the part of the teachers in the study was outside their contractual day which was extremely difficult to manage with the personal and professional demands of each of the individuals.

As Commeyras (1995) stated, "if we take over all or most of the questioning, we miss out on learning with our students, and we deny them the experiences they need to hone their questioning ability" (p.105). An inquiry based approach to teaching does not always need to be about discovery, but rather "an approach to education [that] privileges students' natural questions, and their questions become the center of teaching and learning experiences" (Commeyras, p. 105).
References


Questioning Strategies


Today, 23(5), 14-22.


Appendix A

Lesson 1: Introduction to Imaginary Numbers

Name ___________________________  Date ______
Unit 5 Day 1  Math B

**Intro to Imaginary Numbers**

Solve the following quadratic equation using the quadratic formula.
\[ x^2 - 10x + 40 = 0 \]

The solution of the equation requires \( \sqrt{-15} \) to be a number. But -15 does not have a square root that is a real number. To overcome this problem, the \( i \) number was invented and was defined as follows:
\[ i = \sqrt{-1} \]

Radicals can be "reduced" into a form including \( i \).

Ex.) \( \sqrt{-9} \) \( \sqrt{-25} \) \( \sqrt{-50} \)

When operations are performed on the square roots of negative numbers, the first step in reducing is pulling out the \( i \) !!!

Simplify and Combine each of the following:

a.) \( \sqrt{-49} - 2\sqrt{-4} \)  

b.) \( 3\sqrt{-2} + \sqrt{-8} \)

c.) \( \sqrt{-48} + \sqrt{-27} \)  

d.) \( 4\sqrt{-5} + 3\sqrt{-20} - 6\sqrt{-45} \)
Teacher 1: Period 4

T = Teacher, S = Student

T: Who can share their solution to this problem: $x^2 - 10x + 40 = 0$?

T: When you solved the quadratic, was it factorable?

S: No you needed the quadratic formula.

T: Did you get $\sqrt{-15}$ somewhere in your answer?

S: $\sqrt{4} \sqrt{15}$

T: Why not simplify this to get $\sqrt{-60}$? Do I know $\frac{\sqrt{4} \pm \sqrt{4} \sqrt{-15}}{2} = \frac{10 \pm 2\sqrt{-15}}{2}$

T: Could I do one more step?

S: Divide everything by 2.

T: How can that not be a solution? Put it in the calculator. What can you imagine the parabola doing?

S: It won’t hit the axis.

T: It’s an imaginary number for a non-real root. What piece is causing the problem?

S: The negative part.

T: What are the factors of $\sqrt{-9}$?

S: -3.

S: 9 and -1

T: Do I know $\sqrt{9}$?

S: 3
T: Get $\sqrt{9} \cdot \sqrt{-1}$. What is $\sqrt{-1}$ to?

S: $i$

T: $\sqrt{-9}$ can be expressed as $3i$.

T: How about $\sqrt{-25}$?

S: $5i$

T: What steps did you do?

S: $\sqrt{25} \sqrt{2} \sqrt{-1}$

T: What is $\sqrt{25} \times 5$?

T: What can I do to isolate the factors in terms of $i$?

T: Why does $5\sqrt{2}i$ look funny?

S: It should be $5i\sqrt{2}$ because it would be hard to tell if the $i$ is under the radical.

T: Yes, let's bring it out with the coefficient.

T: What happens when you square $i$?

S: Can you combine $i$'s?

T: What is $4i + 2i$?

S: $6i$

T: What is $4i \times 2i$?

S: $8$

T: What about the $i$'s?

T: What math is there that I can do if I raise $i$ to different powers? $i^2 = ?$ $i = ?$

S: $-1$

T: $i^2 = ?$

T: $i \times i$?
S: $\sqrt{-1} \times \sqrt{-1}$

T: What is a radical times itself?

S: itself

T: What's underneath the radical. What about $i^2$?

S: $\sqrt{-1} \times \sqrt{-1} \times \sqrt{-1} \times \sqrt{-1}$

S: Aren't they all just $-1$s?

T: What is $i^4$?

S: $-1$, nope wait hang on

T: Use our exponents or rules from exponents

S: $i^2 \times i^2$

T: When we multiply we add exponents

S: $-1 \times -1 = 1$

T: What happens when I get to the $5^{th}$ power?

S: $i^2 \times i^3$

T: What is $i^5$?

S: $-1$

T: What is $i^6$?

T: Have we seen this before? Are we starting to build a pattern?
Teacher 1: Period 9

T = Teacher, S = Student

T: Who can share their solution to this problem: $x^2 - 10x + 40 = 0$?

T: What did you get for the solution?

S: For what? The quadratic formula?

T: What was the value for the right side of the equation?

T: What does it simplify to?

S: 100

T: -4 x 40 x 1?

S: 160

T: You said there is no solution. Why?

S: Because it is the square root of a negative number.

T: Put the equation into the y = on the calculator. Do you get a parabola?

S: Yes.

T: What do you get for $\sqrt{-15}$, from $\sqrt{-60}$?

S: Simplify.

S: It's 2 x 2 x -15.

T: Don't you want to be able to simplify as much as possible?

S: Yes, if you take care of the negative part, you can factor out a 2.

T: What should we do with the square root of the negative 1?

S: Pull it out
S: Don’t you put an i in it?

T: Now we will.

T: What is the piece of the number that is causing the problem?

S: -15

T: \( \sqrt{-15} \) or -15?

S: \( \sqrt{-15} \).

T: What about \( \sqrt{-25} \)?

T: If I had \( \sqrt{-9} \), I do not know the answer to it, but what could I do to evaluate it farther or simplify it further?

T: \( \sqrt{9} \cdot \sqrt{-1} \)

T: What’s \( \sqrt{9} \)?

S: 3

T: What \( \sqrt{-1} \)?

S: \( i \)

T: What is \( \sqrt{-25} \) equal to?

S: 5\( i \)

T: What is the next one? Need a couple of minutes?

S: Yeah.

T: Just go about it the same way.

S: Is there a radical sign in the answer?

T: For this one there is a radical sign, but will there always be one in the answer?

T: What do you think the answer should be?

S: Is the \( i \) under the square root?
T: Excellent question. Class what do you think?

S: It's like an x value, right?

T: Exactly

T: $4i + 2i$ will give you what?

S: $6i$

S: What do we do if we have $x^2$?

T: Excellent question, let's look at that.

T: Can you add $4i + 2i^2$?

S: No

T: What is $i^2$ = ?

S: 1, no wait -1

T: How did you get 1?

S: Never mind, I'd want it to be -1.

T: What is $-1 \times -1$?

S: 1

T: What about $i^3$?

S: $i \times i \times i\ or\ \sqrt{-1} \times \sqrt{-1} \times \sqrt{-1}\ which\ equals...-\sqrt{-1}\ or\ if\ you\ like\ better\ -i.$

T: What about the next power? What should I be doing? What does it equal?

S: 1

T: How did you get it?

S: It is $\sqrt{-1}$ four times.

S: How can you get $i^{2005}$?

T: There is a way to figure it out mathematically or using a pattern. Think about it. We discuss it tomorrow.
Teacher 2: Period 2

T = Teacher, S = Student

T: Please find the solution to the quadratic on the handout.

T: How many people need more time.

T: How are we doing with this?

S: It's $\sqrt{-60}$?

T: If this were a $\sqrt{60}$, how would we break this down?

T: Are there 2 factors where one is a perfect square?

T: Which one would you want to keep +?

S: 4 and 15.

T: Why 4?

T: Why did you say 15?

T: You can’t reduce $\sqrt{-4}$?

T: Can you not have the square root of a negative number?

S: Are these imaginary numbers?

S: Where do you get non-real?

T: Can you break down $\sqrt{4}$ to a 2?
Teacher 3: Period 1

T = Teacher, S = Student

T: Does someone remember what an imaginary number means?

S: i

T: What was i?

S: The square root of negative one.

T: Does everyone have one?

T: If the \( \sqrt{-1} = i \), how does \( \sqrt{-49} \) breakdown?

T: Emily do you want to share?

S: No.

T: \( \sqrt{-49} \) breaks down to \( \sqrt{-1\cdot49} = \) What are these pieces to?

S: 7i

T: What about \( \sqrt{-2} \) ?

S: \( \sqrt{-1}\sqrt{2} = i\sqrt{2} \)

T: How many of you are following so far?

S: (Most raised hands.)

T: What about \( \sqrt{-2} + \sqrt{-8} \) ?

S: Is that right?

T: Almost

T: What does \( \sqrt{-8} \) become?

S: \( 2\sqrt{2}i \)

T: How many of you got this much \( \sqrt{-1\cdot4\cdot\sqrt{2}} \) ?
T: So the last step is \(2\sqrt{2}i\).

S: Does it matter where the \(i\) is?

T: No, but do you think it could be confusing at the end versus the front so it wouldn’t look like it’s under the radical?

T: How long do you think imaginary numbers or \(i\) have been around for?

S: Just by the way you said the question, I don’t think so long.

T: How long have quadratic equations been around?

S: Since ancient Greece.

T: Why do you think ancient Greece?

S: This is when equations were created.

T: What was going on in the 1500’s?

S: The Middle Ages.

S: Beginning of the Renaissance.

T: Right, the end of the Dark Ages.

T: What went on in the Renaissance?

S: Revamped Math and Science.

T: Actually Art and Music. Positive changes and growth. Land owners tried to maximize profits from crops and have to come up with creative calculations for area.

T: \(i\) is what?

S: \(i = \sqrt{-1}\)

T: What is \(i^2\)?

S: \(\sqrt{-1}\sqrt{-1}\) which just becomes \(\sqrt{1}\)
T: Just 1 right?

T: What about $i^3$?

S: -1

T: How?

S: $i^2 \times i$ which is $-1 \times \sqrt{-1}$ oh wait that is just $-\sqrt{-1}$

T: What about $i^4$?

S: -2

S: Isn't it $-\sqrt{-1} \times i$? so wouldn't it be $-\sqrt{-1} \times \sqrt{-1}$ which would be $-1 \times -1 = 1$

T: or $i^2 \times i^2$ or $-1 \times -1 = 1$ which is the same thing right?

T: How many of you are following so far?

S: Many raised their hands.

T: What about $i^5$?

S: $i^5 = i^2 \times i^2 \times i$

T: Which is $-1 \times -1 \times \sqrt{-1}$?

S: Yeah.

T: Is this what you were going to say?

S: Yeah.

T: So $i^6 = i^4 \times i^2$. Why did I pick a four for one of the exponents?

S: Its easier.

T: What is $i^6$ =?

S: 1

T: What is $i^2$?

T: How are we going to break $i^7$ down?
T: Asked another student to help the first student, what do you think?

T: \[ i^7 = i^4 \times i^2 \times i = 1 \times -1 \times i \]

T: Anyone see a pattern yet?

T: \( i^7 \) is the same as what?

S: \( i^3 \)

T: \( i^6 \) is the same as what?

T: class?

S: \( i^2 \)

T: What do you think \( i^8 \) is = to?

S: \[ i^8 = i^4 \times i^4 = 1 \times 1 = 1 \]

T: How many of you followed this?

T: Homework – finish both worksheets

T: Ticket out the door – no time to write responses - give me 1 main idea and 1 question from today’s class, anyone. (Bell rang).
Appendix C

Review Lesson Investigation 1 Triangulation

INVESTIGATION 1: Triangulation

One of the most effective strategies for calculating distances to points that cannot be reached is to model the situation with a triangle in which the segment of unknown length is one side and other parts can be measured. If you can make the model a right triangle, there are several ways to calculate the unknown length.

Study the diagram below of a radio transmission tower with two support wires attached to it and to the ground.

1. First focus on the triangle formed by the tower, the ground, and the shorter support wire.
   a. If \( CB = 50 \) feet and \( BD = 100 \) feet, how long is the support wire \( CD \)?
   b. If you were asked to attach a support wire, 125 feet long, from point \( C \) to the tower, how high up the tower would you have to climb?
   c. What general relationship among sides of a right triangle have you used in answering Parts a and b?
   d. Refer to the diagram at the right. What calculations are required to find a missing length in a right triangle when the lengths of both legs \( (r\) and \( p\) are known? What if the length of one leg and the hypotenuse \( q \) and \( q\) are known?

2. Next, focus on the triangle formed by the tower, the ground, and the longer support wire \( AE \). Length \( AB = 75 \) feet and \( \angle BAE = 60^\circ \). (The notation used is read the measure of angle...).
   a. How long is the support wire from \( A \) to \( E \)?
   b. How high up the tower is point \( E \), where the support wire from \( A \) is attached?
   c. What is the degree measure of \( \angle ABE \)?
   d. How could you use the measure of \( \angle ABE \) and the length of \( AB \) to calculate the lengths of \( EB \) and \( EA \)?

3. Refer to your work in Part a of Activity 1. Use the information to estimate the measures of \( \angle BCD \) and \( \angle BDC \).
4. Think again about the problem of finding the height above sea level of Mount Everest or of other very tall structures. What measurements would allow you to calculate heights that cannot be measured directly? What mathematical principles and relationships would you need to make the calculations?

**Checkpoint**

The activities in investigation 1 required combinations of facts about several parts of right triangles to find information about other parts.

1. What general principle relates the lengths of the sides of a right triangle? How can this relationship be used to calculate the length of one side when the other two are known?

2. What general relationships connect side and angle measurements in any right triangle? How can these relationships be used to calculate the unknown length of one side when another side length and angle measurement are given?

*Be prepared to explain these right triangle relationships and their use to your classmates.*

**On Your Own**

The following sketch shows the start of one surveyor's attempt to determine the height of a hill without climbing to the top itself.

![Sketch of a hill and a surveyor's attempt to measure its height](image)

- Use the given information to calculate the lengths of AB and BC.

Suppose that a laser ranging device allowed you to find the length of AB and the angle of elevation \( \angle BAC \), but you could not measure the length of BC. How would you use this information instead of the information from the diagram to calculate the lengths of AC and BC?
Lesson 2 Investigation 2 The Law of Sines

INVESTIGATION 2 The Law of Sines

As you worked on the activities in Investigation 1, you found several ways the Pythagorean Theorem and the trigonometric ratios sine, cosine, and tangent could be used to calculate unknown side lengths of right triangles. If the triangle isn’t a right triangle, it's not so easy to model the situation involving the unknown distance, but it can be done.

For example, suppose that two park rangers who are in towers 10 miles apart in a mountain forest spot a fire that is uphill and far away from both. Suppose that one ranger recognizes the fire location and knows that it is about 4.9 miles from that tower. With this information and the angles given in the diagram below, the rangers can calculate the distance of the fire from the other tower.

One way to start working on this problem is to divide the oblique triangle into two right triangles as shown below.

At first this doesn’t seem to help much. Indeed, all two segments of an angle length there are now four! On the other hand, there are now three triangles which you can seek useful relationships among the known sides and angles.
1. Use trigonometry or the Pythagorean relationship for right triangles to find the length of BC. When you have one sequence of calculations that gives the desired result, see if you can find a different approach.

2. In one Maryland class, a group presented their solution to Activity 1 and claimed that it was the cleanest, quickest method possible. Check each step in their reasoning and explain why each step is or is not correct:

   \[(1) \, \frac{h}{a} = \sin \angle B \]
   \[(2) \, h = a \sin \angle B \]
   \[(3) \, 4.0 = 4.0 \sin \angle B \]
   \[(4) \, \arcsin \left( \frac{4.0}{4.0} \right) = \angle B \]
   \[(5) \, \angle B = 90.0^\circ \]
   \[(6) \, h = 4.0 \, \text{ft} \]
   \[(7) \, h = 4.0 \, \text{ft} \]

   Comparing your solution with this reported solution.

3. The approach in Activity 2 to calculating the unknown side length of a triangle that is not itself a right triangle illustrates a very useful general relationship among sides and angles of any triangle. Explain why each step in the following derivation is correct.

\[(1) \, \frac{h}{a} = \sin \angle A \]
\[(2) \, h = a \sin \angle A \]
\[(3) \, \frac{h}{b} = \sin \angle B \]
\[(4) \, h = b \sin \angle B \]
\[(5) \, h = a \sin \angle B \]
\[(6) \, \frac{h}{c} = \sin \angle C \]
\[(7) \, h = c \sin \angle C \]

The relationship derived in Activity 3 holds in any triangle and for all three sides and angles as well. It is called the Law of Sines and can be written in two equivalent forms:

- **In any triangle** \( \triangle ABC \) with sides of lengths \( a, b, \) and \( c \) opposite \( \angle A, \angle B, \) and \( \angle C, \) respectively:

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]

- **Equivalent forms:**

\[
\frac{a}{b} = \frac{\sin A}{\sin B} \quad \text{and} \quad \frac{b}{c} = \frac{\sin B}{\sin C}
\]
You can use this important relationship to calculate measures of angles or lengths of sides in triangles with even less given information than the fire spotting problem at the beginning of this investigation. In practice, you only need to use the equality of two of the ratios at any one time.

4. Suppose that the two rangerr spot another fire in a different spot, indicated in the next diagram. Use what you know about angles in a triangle and the Law of Sines to find the distances from each tower to the fire.

```
Fire

A
135

C

D
E

B
10 miles

```

5. In parallelogram $ABCD$, information is given about one side and two angles formed by the diagonals $AC$ and $BD$.

```
D

E

B

5 cm

```
a. Recall from your previous study of mathematics that a parallelogram has 180° rotational symmetry about the point of intersection of its diagonals. Verify this fact by tracing parallelogram $ABCD$ on a sheet of paper and rotating the paper about point $E$.

b. Use facts about triangles and the rotational symmetry property of parallelograms to find as much additional information as you can about the other 7 segments and 10 angles in the given figure. Do not use trigonometry.

c. Use the Law of Sines to find further information about the segments and angles in the figure.
Checkpoint

The Law of Sines states a relation among sides and angles of any triangle. It can be used to find unknown side lengths or angle measures from given information. Suppose you have modeled a situation with \( \triangle PQR \) as shown below.

1. What combinations of information about the sides and angles of \( \triangle PQR \) will allow you to find the length of \( QR \)? How would you use that information to calculate \( QR \)?

2. What combinations of information about the sides and angles of \( \triangle PQR \) will allow you to find the measure of \( \angle Q \)? How would you use that information to calculate \( \angle Q \)?

Be prepared to explain your thinking to the entire class.

On Your Own

A commuter airplane set course over the Atlantic Ocean, reported experiencing mechanical problems around 9:15 pm. The pilot sent two calls, one to Boston Logan International Airport and one to the regional airport in nearby Beverly. Air traffic controllers at the two airports reported the angles shown in the diagram below. How far was the plane from the closer airport?
Appendix E

Homework 1 Law of Sines Student Work

\[ \sin 26^\circ = \sin 73^\circ \]
\[ 25 \times \sin 26^\circ = 25 \times \sin 73^\circ \]
\[ y = 54,5375916 \text{ km} \]

The plane is about 55 km away from the closest airport.

Which airport should the plane land at?
The Law of Sines states a relation among sides and angles of any triangle. It can be used to find unknown side lengths or angle measures from given information. Suppose you have modeled a problem situation with $\triangle PQR$ as shown below.

a. What combinations of information about the sides and angles of $\triangle PQR$ will allow you to find the length of $QR$? How would you use that information to calculate $QR$?

b. What combinations of information about the sides and angles of $\triangle PQR$ will allow you to find the measure of $\angle Q$? How would you use that information to calculate $m\angle Q$?

Be prepared to explain your thinking to the entire class.
The Law of Sines states a relation among sides and angles of any triangle. It can be used to find unknown side lengths or angle measures from given information. Suppose you have modeled a problem situation with $\triangle PQR$ as shown below.

**a** What combinations of information about the sides and angles of $\triangle PQR$ will allow you to find the length of $QR$? How would you use that information to calculate $QR$?

**b** What combinations of information about the sides and angles of $\triangle PQR$ will allow you to find the measure of $\angle Q$? How would you use that information to calculate $m\angle Q$?

*Be prepared to explain your thinking to the entire class.*
with a corresponding side,

\[
\frac{\sin \theta}{\frac{e}{r}} = \frac{\sin \alpha}{\frac{p}{r}}
\]

\[
\frac{\sin \alpha}{\frac{p}{r}} - \frac{\sin \theta}{\frac{e}{r}} = 0
\]
Appendix G

Homework 2 Law of Sines Student Work

Use the Law of Sines to solve the following questions. Show all work.

1) In \( \triangle ABC \), \( \angle A = 72^\circ \), \( \angle C = 43^\circ \), and \( a = 25 \). Draw a labeled diagram and then find \( b \) and \( c \) to the nearest integer.

2) In the figure, two observers at points A and C, 8 km apart, sight a boat at the same instant. How far (to the nearest kilometer) is the boat from the farthest observer?

3) Find the length of the indicated side to the nearest integer.

4) In \( \triangle ABC \), \( AB = 81 \) feet, \( \angle A = 61^\circ \), and \( \angle C = 73^\circ \). Find the length of \( AC \) correct to the nearest foot.
Show all work.

1) Suppose two defenders who are 15 feet apart spot the opposing quarterback frozen with no one to throw the ball to. Use this information and the angles given in the diagram below to find out how far (to the nearest tenth) each defender has to go to sack the quarterback. Then state who was closer and by how much.

![Diagram of a triangle with sides and angles labeled]

Given a triangle with $a = 15$, $b = 21$, and $C = 10^\circ$, what is the length of $c$?

Round the answer to the nearest decimal place.

In $\triangle ABC$, $m \angle A = 33$, $a = 12$, and $b = 15$.

What is $\angle C$ to the nearest degree?
Show all work.

1) Suppose two defenders who are 15 feet apart spot the opposing quarterback frozen with no one to throw the ball to. Use this information and the angles given in the diagram below to find out how far (to the nearest tenth) each defender has to go to sack the quarterback. Then state who was closer and by how much.

[Diagram of a triangle showing defenders and a quarterback]

- Let $x = \text{def. B's distance}$
- Let $y = \text{def. A's distance}$

Given a triangle with $\angle A = 15\degree$, $\angle B = 64\degree$, and $\angle C = 101\degree$. Find the length of $c$.

Round the answer to two decimal places.

- $\sin 143\degree = \sin 31\degree$
- $\sin 143\degree = 15 \cdot \sin 31\degree$
- $c = 15 \cdot \sin 31\degree$
- $c = 8.43$

In $\triangle ABC$, $a = 12$, $c = 15$, and $b = 13$.

What is $a$ to the nearest integer?

- $\sin 33\degree = \sin B$
- $\frac{12}{15}$
- $\frac{12}{15} = 12 \cdot \sin 33\degree$
- $\sin 33\degree = 6.93$
- $\sin 33\degree = 0.56$
- $\sin 33\degree = 0.56$
- $a = 4.97$

Find the length of $b$ (over).
INVESTIGATION 3 The Law of Cosines

For all right triangles, the Pythagorean Theorem shows how the lengths of the two legs and the hypotenuse are related to each other. When that relationship is expressed as an equation, it is possible to solve for any one of the variables in terms of the others. It's natural to wonder what is so special about right triangles and how the relationship among the sides changes as the right angle changes to an acute or obtuse angle.

1. Consider a linkage with two sides of fixed length 12 cm and 16 cm. Here, \( AC = 12 \text{ cm} \) and \( BC = 16 \text{ cm} \).
   a. What is the distance from \( A \) to \( B \) when the angle at \( C \) is a right angle?
   b. How does the distance from \( A \) to \( B \) change as \( \angle C \) is rotated to make smaller and smaller angles at \( C \)? How does the distance change if \( \angle C \) is rotated to make larger angles at \( C \)?

2. Using an actual physical linkage or careful drawings, test your answers to Activity 1 by carefully measuring the distance from point \( A \) to point \( B \) at these cases:
   a. \( m \angle C = 30^\circ \)
   b. \( m \angle C = 70^\circ \)
   c. \( m \angle C = 130^\circ \)
   d. \( m \angle C = 150^\circ \)

3. Why is it impossible to check the measured distances in Activity 2 by calculations using the Law of Sines, without getting more information?

There is a second trigonometric principle for finding relationships among side lengths and angle measures of any triangle. It is called the Law of Cosines.

In any triangle \( ABC \) with sides of lengths \( a \), \( b \), and \( c \), and opposites \( \angle A \), \( \angle B \), and \( \angle C \), respectively,

\[
c^2 = a^2 + b^2 - 2ab \cos C
\]

This is another of these very useful equations that link several geometric variables.
1. The Law of Cosines states a relation among the lengths of three sides of a triangle and the cosine of one angle of the triangle. If you know the lengths of two sides and the measure of the angle between the two sides, you can calculate the length of the third side.

a. State in words how you would calculate the length $c$ in $\triangle ABC$ if you know $a$, $b$, and the measure of $\angle C$.

b. Write the Law of Cosines to calculate the length $c$ in $\triangle ABC$ if you know $a$, $b$, and the measure of $\angle C$.

c. Write a third form of the Law of Cosines for $\triangle ABC$, for when you know the measure of $\angle A$.

d. Suppose in $\triangle PQR$ you needed to calculate the length of $QR$. What information would you need in order to use the Law of Cosines? Write the equation you would use.

e. Suppose in $\triangle PQR$ you know $\angle P$, $\angle Q$, and the lengths $P$ and $Q$. Could you find the length $R$ using the Law of Cosines? Explain your reasoning.

f. Using the information given in Part e, how could you find the length $R$ using the Law of Sines? Which method would you prefer to use? Why?

5. Surveyors often are faced with irregular polygonal regions for which they are asked to locate and stake out boundaries, determine elevations, and estimate areas. Some of these tasks can be accomplished by using a site map with data similar to the plan shown in the figure below. In one subdivision of property near a mid-size city, a plot of land had the shape and dimensions shown.
Examine the triangulation of the plot shown below.

a. How can the Law of Cosines be used to find AC?

b. Find AC to the nearest tenth of a meter.

c. How can the Law of Cosines be used to find AD?

d. Find AD to the nearest tenth of a meter.

6. The Law of Cosines, \( c^2 = a^2 + b^2 - 2ab\cos(C) \), states a relation among the lengths of three sides of a triangle ABC and the cosine of an angle of the triangle. If you know the lengths of all three sides of a triangle, you can calculate the cosine of an angle and then determine the measure of the angle itself.

a. Solve the equation \( c^2 = a^2 + b^2 - 2ab\cos(C) \) for \( \cos(C) \).

b. Using your results from Activity 5, find the measure of \( \angle ABC \) to the nearest tenth of a degree. What is the measure of the third angle in \( \triangle ABC \)?

c. Now find the area of \( \triangle ABC \).

d. Explain how you could determine the area of the entire triangular plot.

7. Now that you understand how to use the Law of Cosines, examine more closely its symbolic form and the information it conveys. Consider again the linkage with arms of lengths 12 cm and 16 cm positioned at various possible angles.

a. Record your data from Activity 2 in a copy of the table below. Then, using a physical linkage or careful drawings, complete the remainder of your table, showing how the distance from the end of one arm to the other changes as the angle at link point C changes.

<table>
<thead>
<tr>
<th>( \angle C ) (degrees)</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>90</th>
<th>120</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length ( AB ) (cm)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**UNIT 1: MULTIPLE VARIABLE MODELS**
2. Now add a row to your table from Part a showing corresponding values of
lengths $AB$.

<table>
<thead>
<tr>
<th>Length</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$AC$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**c.** What is $\angle C$ when $m\angle C = 90^\circ$? and how does that simplify the equation for the Law of Cosines?

**d.** In what sense does the term "$AC \cdot C$" act as a correction term, adjusting the Pythagorean relationship for triangles in which $\angle C$ is not a right angle?

8. As you have seen, the Law of Sines and the Law of Cosines can be used to find the measures of unknown angles and sides. You have already given information about side and angle measurements to decide which law to apply. Then you have to work with the resulting equations to solve for the unknown angle or side measurements.

For example, suppose that two sides, $BC$ and $AC$, and a diagonal $AC$ of a parallelogram $ABCD$ measure 7 cm, 9 cm, and 11 cm respectively.

**a.** Draw and label a sketch of such a figure.

**b.** Which of the two trigonometric laws can be used to find the measure of an angle in that parallelogram?

**c.** Find the measure of an angle to the nearest tenth of a degree.

**d.** The diagonal $AC$ splits the parallelogram $ABCD$ into two triangles. Find the remaining measures of the angles in those triangles.

**e.** Find the length of diagonal $BD$.

**Checkpoint**

Consider $\triangle ABC$ shown at the right.

1. What information would you need to know in order to use the Law of Cosines to find the length of $AC$? What equation would you use to find that length?

2. What information would you need to know in order to use the Law of Cosines to find the measure of $\angle A$? What equation would you use to find that angle measure?

3. Suppose you know the lengths $a$, $b$, and $c$. What can you conclude about $m\angle B$ if $a^2 + b^2 > c^2$? If $a^2 + b^2 < c^2$? If $a^2 + b^2 = c^2$?

Be prepared to explain your thinking to the entire class.
Revised Lesson 3 Law of Cosines

Name ____________________________ Date __________
Unit 6 Math B

Law of Cosines

So far you have only used trigonometry in _________ triangles. By using a form of the distance formula, we can use trigonometry in any triangle to find the measure of sides and angles.

The following formula is called The Law of Cosines.

\[ a^2 = b^2 + c^2 - 2bc \cos A \]

It can be used to find the third side of a triangle if you know the two other sides and the angle in between.

Example:

\[ \text{In } \triangle ABC, \quad b = 11, \quad c = 12, \quad \text{and } m < A = 120^\circ. \quad \text{Find } a \text{ to the nearest integer.} \]

(Be sure to remember order of operations!!!)

It is important to know that any side of the triangle could be found using the Law of Cosines. To accommodate this the formula can be written 3 different ways:

\[ a^2 = b^2 + c^2 - 2bc \cos A \]
\[ b^2 = a^2 + c^2 - 2ac \cos B \]
\[ c^2 = a^2 + b^2 - 2ab \cos C \]
Practice:
1) In $\triangle ABC$, $a = 2$, $b = 5$, and $\cos C = \frac{1}{2}$. The length of side $c$ is
   a) $\sqrt{19}$  b) $\sqrt{20}$  c) $\sqrt{34}$  d) $\sqrt{30}$

2) In $\triangle ABC$, $a = 6$, $b = 8$, and $\cos C = \frac{3}{8}$. What is the length of side $c$?
   a) $\sqrt{118}$  b) $\sqrt{82}$  c) $10$  d) $8$

3) In $\triangle ABC$, $a = 3$, $b = 5$, and $m \angle C = 120^\circ$. Find the value of $c$.

4) If the lengths of two sides of a triangle are 7 and 10 and the cosine of the included angle is $-\frac{1}{7}$, what is the length of the third side?

5) In $\triangle BCD$, $b = 3\sqrt{3}$, $a = 6$, and $m \angle D = 30^\circ$. Find $d$.

6) Find to the nearest integer the measure of the base of an isosceles triangle if the measure of the vertex angle is 84 degrees and the measure of each leg is 12.
The Law of Cosines can also be used to find one of the angles if all 3 sides are known.

Example:

Find the measure of the largest angle, to the nearest degree, of the above triangle if the measures of the sides of the triangle are 5, 6, and 7.

Practice:

1) In \( \triangle ABC \), \( b = 9 \), \( c = 12 \), and \( d = 15 \). The cosine of the angle \( A \) is

   (1) 1  
   (2) 0  
   (3) \( \frac{4}{5} \)  
   (4) 90

2) In \( \triangle XYZ \), \( x = 5 \), \( y = 7 \), and \( z = 8 \). Find \( m \angle X \).

3) In \( \triangle ABC \), the measures of the sides are 3, 5, and 7. Find the measure of the smallest angle in the triangle.
Appendix K

Mixed Review Law of Sines and Cosines

Use the Law of Sines and/or the Law of Cosines to solve the following word problems:

1. The diagonals of a parallelogram are respectively 7.0 inches and 10.0 inches long, and they intersect at an angle of 65°. Find the length of one of the longer sides of the parallelogram to the nearest tenth of an inch.

2. A canoe race is to be run over a triangular course marked by buoys A, B, and C. The distance between A and B is 100 yards, that between B and C is 160 yards, and that between C and A is 220 yards. Find, to the nearest degree, m∠ABC.
2. Two straight roads \( \overline{RT} \) and \( \overline{ST} \) intersect at a town \( T \) and form with each other an acute angle of \( 67^\circ \). Towns at \( R \) and \( S \) are 22 miles and 31 miles respectively from \( T \). Find to the nearest mile the distance between towns \( R \) and \( S \).

4. A surveyor on the ground takes two readings of the angle of elevation of the top of a tower. From the 150 ft apart, the measures are \( 50^\circ \) and \( 70^\circ \). Find the tower's height to the nearest foot.
Appendix L

Review for Quiz Law of Sines, Cosines, Basic Right Triangle Trigonometry

Name: ___________________________ Date: ___________________________
Math B Review for Quiz #3

This packet will NOT tell you what to use, however, you will need to use either Law of Sines, Cosines, or Special Right Triangle Ratios.

Solve the various trigonometric "puzzles" by deciding how to use the information that is provided, with the information that is needed for the various laws.

Use separate pieces of paper if needed.

The formula for the Law of Sines is...

The formula for Law of Cosines is...
(Write all three versions)

The three special ratios for a right triangle are:

1. In \triangle ABC, a = 19, b = 10, and \angle A = 111. Which statement can be used to find the value of \angle C?

   (1) \sin C = \frac{10}{19} \\
   (2) \sin C = \frac{19\sin69^\circ}{10} \\
   (3) \sin C = \frac{10\sin21^\circ}{19} \\
   (4) \sin C = \frac{10\sin69^\circ}{19}

2. In \triangle ABC, \angle A = 53, a = 12, and b = 15. What is \angle B to the nearest degree?
   (1) 41 \\
   (2) 43 \\
   (3) 44 \\
   (4) 48
3. To measure the distance through a mountain for a proposed tunnel, surveyors chose points A and B at each end of the proposed tunnel and a point C near the mountain. They determined that AC = 3,800 meters, BC = 2,900 meters, and \( \angle ACB = 110 \). Draw a diagram to illustrate this situation and find the length of the tunnel, to the nearest meter.

4. In the accompanying diagram of right triangle \( ABC \), \( AB = 5 \), \( BC = 15 \), \( AC = 17 \), and \( \angle ABC = 90 \).

What is \( \tan \angle ACB \)?

(1) \( \frac{8}{15} \)  
(2) \( \frac{17}{15} \)  
(3) \( \frac{8}{17} \)  
(4) \( \frac{15}{17} \)

5. In \( \triangle ABC \), \( \angle A = 53 \), \( \angle B = 14 \), and \( a = 10 \). Find \( b \) to the nearest integer.

6. A ship on the ocean surface detects a sunken ship on the ocean floor at an angle of depression of 50°. The distance between the ship on the surface and the sunken ship on the ocean floor is 200 meters. If the ocean floor is level in this area, how far above the ocean floor, to the nearest meter, is the ship on the surface?
7. A ski lift begins at ground level 0.75 mile from the base of a mountain whose face has a 50° angle of elevation, as shown in the accompanying diagram. The ski lift ascends in a straight line at an angle of 20°. Find the length of the ski lift from the beginning of the ski lift to the top of the mountain, to the nearest hundredth of a mile.

[Diagram of a ski lift]

8. A wooden frame is to be constructed in the form of an isosceles trapezoid, with diagonals acting as braces to strengthen the frame. The sides of the frame each measure 5.30 feet, and the longer base measures 12.70 feet. If the angles between the sides and the longer base each measure 68.4°, find the length of one brace to the nearest tenth of a foot.

[Diagram of an isosceles trapezoid]

9. A ship at sea heads directly toward a cliff on the shoreline. The accompanying diagram shows the top of the cliff, D, sighted from two locations, A and B, separated by distance S. If $m_{\angle DAC} = 30^\circ$, $m_{\angle DBC} = 45^\circ$, and $S = 30$ feet, what is the height of the cliff, to the nearest foot?
10. In the accompanying diagram of $m \angle h = 70^\circ$, $m \angle A = 65^\circ$, $m \angle B = 70^\circ$, and the side opposite vertex $B$ is 7, find the length of the side opposite vertex $A$.

11. A ladder leaning against a building makes an angle of $58^\circ$ with level ground. If the distance from the foot of the ladder to the building is 6 feet, find, to the nearest foot, how far up the building the ladder will reach.

12. In $\triangle ABC$, $\sin A = \frac{4}{5}$, $\angle B = 17^\circ$, and $a = 24$. Find the length of side $b$.

13. In $\triangle ABC$, $a = \sqrt{2}, b = 3$, and $m \angle B = 45^\circ$. Find the value of $\sin A$. 
14. As seen in the accompanying diagram, a person can travel from New York City to Buffalo by going north 170 miles to Albany and then west 280 miles to Buffalo.

![Diagram showing the route from New York City to Buffalo via Albany.]

a. If an engineer wants to design a highway to connect New York City directly to Buffalo, at what angle, \( x \), would she need to build the highway? Find the angle to the nearest degree.

b. To the nearest mile, how many miles would be saved by traveling directly from New York City to Buffalo rather than by traveling first to Albany and then to Buffalo?
Appendix M

Quiz 3 Law of Sines, Cosines and Basic Right Triangle Trigonometry

Student 1

unit 6: Trigonometry - Quiz #3

1. In the accompanying diagram of \( \triangle ABC \), \( m\angle A = 80^\circ \), \( m\angle C = 50^\circ \) and, \( AC = 13 \)

\[
\frac{\sin B}{b} = \frac{\sin C}{c}
\]

\[
\sin 100^\circ = \sin 50^\circ \]

What is the length of side \( AB \) to the nearest tenth?

(1) 6.6
(2) 10.1
(3) 11.5
(4) 12.0

2. Which ratio represents \( \cos A \) in the accompanying diagram of \( \triangle ABC \)?

\[
\cos = \frac{b}{c}
\]

(1) \( \frac{5}{13} \)
(2) \( \frac{12}{13} \)
(3) \( \frac{5}{12} \)
(4) \( \frac{12}{5} \)

3. The angle of elevation from a point 25 feet from the base of a tree on level ground to the top of the tree is \( 30^\circ \). Which equation can be used to find the height of the tree?

(1) \( \tan 30^\circ = \frac{x}{25} \)
(2) \( \cos 30^\circ = \frac{25}{x} \)
(3) \( \sin 30^\circ = \frac{x}{25} \)
(4) \( 30^\circ \times 25 = x \)
4. A person standing on level ground is 2,000 feet away from the foot of a 420-foot-tall building, as shown in the accompanying diagram. To the nearest degree, what is the value of $x$?

![Diagram showing a triangle with sides and angles labeled.]

\[ \tan x = \frac{420}{2000} \]

\[ \tan^{-1} \left( \frac{420}{2000} \right) \approx 12^\circ \]

5. As shown in the accompanying diagram, two tracking stations, A and B, are on an east-west line 110 miles apart. A forest fire is located at F, on a bearing $42^\circ$ northeast of station A and $15^\circ$ northeast of station B. How far, to the nearest mile, is the fire from station A?

![Diagram showing two triangles with labeled sides and angles.]

\[ \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \]

\[ \frac{\sin 110^\circ}{110} = \frac{\sin 27^\circ}{b} \]

\[ b = 234 \text{ miles} \]

6. Two straight roads, Elm Street and Pine Street, intersect creating a $40^\circ$ angle, as shown in the accompanying diagram. John's house (J) is on Elm Street and is 3.2 miles from the point of intersection. Mary's house (M) is on Pine Street and is 5.5 miles from the point of intersection. Find, to the nearest tenth of a mile, the direct distance between the two houses.

![Diagram showing two triangles with labeled sides and angles.]

\[ c^2 = a^2 + b^2 - 2ab \cos C \]

\[ c^2 = 3.2^2 + 5.5^2 - 2 \times 3.2 \times 5.5 \cos 40^\circ \]

\[ c \approx 4.2 \text{ miles} \]
7. The Vietnam Veterans Memorial in Washington, D.C., is made up of two walls, each 246.75 feet long, that meet at an angle of 125.2°. Find, to the nearest foot, the distance between the ends of the walls that do not meet.

\[
\begin{align*}
\sin A &= \frac{\sin B}{b} \\
\sin 27.41 &= \frac{\sin 125.2}{246.75} \\
\sin 27.41 &= \sin 125.2 \times 246.75 \\
\sin 27.41 &= 201.6305 \\
\sin 27 &= \sin 27 \\
D &= 141/4 \text{ feet}
\end{align*}
\]
1. In the accompanying diagram of $\triangle ABC$, $m \angle A = 30^\circ$, $m \angle C = 50^\circ$, and $AC = 13$.

What is the length of side $AB$ to the nearest tenth?

(1) 6.6   (2) 10.1   (3) 11.5   (4) 12.0

2. Which ratio represents $\cos A$ in the accompanying diagram of $\triangle ABC$?

(1) $\frac{8}{13}$   (2) $\frac{12}{13}$   (3) $\frac{13}{8}$   (4) $\frac{12}{5}$

3. The angle of elevation from a point 25 feet from the base of a tree on level ground to the top of the tree is $30^\circ$. Which equation can be used to find the height of the tree?

(1) $\tan 30^\circ = \frac{x}{25}$   (2) $\cos 30^\circ = \frac{x}{25}$

(3) $\sin 30^\circ = \frac{x}{25}$   (4) $30^2 + 25^2 = x^2$
4. A person standing on level ground is 2,000 feet away from the foot of a 420-foot-tall building, as shown in the accompanying diagram. To the nearest degree, what is the value of $x$?

5. As shown in the accompanying diagram, two tracking stations, $A$ and $B$, are on an east-west line 110 miles apart. A forest fire is located at $F$, on a bearing $42^\circ$ northeast of station $A$ and $15^\circ$ northeast of station $B$. How far, to the nearest mile, is the fire from station $A$?

6. Two straight roads, Elm Street and Pine Street, intersect creating a $40^\circ$ angle, as shown in the accompanying diagram. John's house ($J$) is on Elm Street and is 3.2 miles from the point of intersection. Mary's house ($M$) is on Pine Street and is 5.6 miles from the intersection. Find, to the nearest tenth of a mile, the direct distance between the two houses.
7. The Vietnam Veterans Memorial in Washington, D.C., is made up of two walls, each 246.75 feet long, that meet at an angle of 125.2°. Find, to the nearest foot, the distance between the ends of the walls that do not meet.

![Diagram of the Vietnam Veterans Memorial](https://example.com/vietnam-memorial-diagram.png)

\[ c^2 = a^2 + b^2 - 2ab\cos\theta \]

\[ c^2 = 246.75^2 + 246.75^2 - 2(246.75)(246.75)\cos(125.2°) \]

\[ c^2 = 123.71 + 123.71 - 2(246.75)(246.75)\cos(125.2°) \]

\[ c^2 = 123.71 + 123.71 - 191963.9266 \]

\[ c^2 = 123.71 + 123.71 - 191963.9266 \]

\[ c = 438 \text{ feet} \]