Lossless propagation of optical pulses through N-level systems with SU(2) symmetry

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Abstract
Propagation of optical pulses through atomic media consisting of atoms with N transition levels and possessing the so-called SU(2) symmetry is studied. It is shown that there are generally N - 1 sets of conditions, each of which, when satisfied, would permit the appropriate Maxwell-Bloch equations to have a solution having the form of simultaneous different-wavelength optical solutions, so-called simulations. The first two sets of solutions were known previously, but the remaining sets are new.

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Lossless propagation of optical pulses through \(N\)-level systems with SU(2) symmetry

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Propagation of optical pulses through atomic media consisting of atoms with \(N\) transition levels and possessing the so-called SU(2) symmetry is studied. It is shown that there are generally \(N - 1\) sets of conditions, each of which, when satisfied, would permit the appropriate Maxwell-Bloch equations to have a solution having the form of simultaneous different-wavelength optical solitons, so-called simultons. The first two sets of solutions were known previously, but the remaining sets are new.

The concept of lossless propagation of two or more simultaneous (equal-velocity) optical solitons (simultons) that may have widely different wavelengths was introduced by Konopnicki et al.\(^1,2\). A simulton, because of its multiwavelength capability, is different from pulse trains in two-level absorbers\(^3\) and from two-photon self-induced transparency\(^4\). Simulton propagation generally requires several conditions: the \(N\) dipole-connected energy levels (in which the dipoles connect levels \(n\) and \(n + 1\) for \(n = 1, 2, \ldots, N - 1\)) of the atomic medium through which the \(N - 1\) pulses of the simultons propagate must have energies that are ordered in a certain way; the absorbing medium must be partially excited out of its ground state in accordance with appropriate initial conditions; the pulse amplitudes have to satisfy appropriate relations.

Following the work of Konopnicki et al., we extended their result by allowing nonzero equal-one-photon detunings in the \(N - 1\) allowed transitions\(^5\) and also discovered that a different set of conditions on the energy-level configuration, initial-level population distribution, and strengths of the dipole matrix elements would enable an \(N\)-level system to support the propagation of a different set of simultons.\(^6\) The two different sets of conditions that permitted the simulton propagation through an \(N\)-level system were later observed to be mutually exclusive and orthogonal.\(^7\) The question was raised but not answered in Ref. 7 of whether other sets of conditions exist that would permit simulton propagation. For a system of \(N\)-level atoms possessing the so-called SU(2) dynamic symmetry,\(^8-10\) the problem has now been solved in its entirety, and it is the purpose of this paper to report this solution.

The principal result that I shall present is that there are \(N - 1\) sets of mutually exclusive or orthogonal conditions and that, if any one of these conditions is satisfied, it would allow the atomic medium consisting of these \(N\)-level atoms to support the propagation of simultons. The \(N - 1\) sets of conditions will be labeled \(k_1 = 1, 2, \ldots, N - 1\), and the precise conditions will be expressed, in a compact form, in terms of the Wigner 3-\(j\) symbols.\(^11\) The two different sets of conditions discovered previously by Konopnicki et al.\(^1,2\) and by me\(^7\) will be seen to correspond to the sets characterized by \(k_1 = 1\) and \(k_1 = 2\), respectively.

We assume a plane-wave incident electric field \(E(z, t)\) with \(N - 1\) distinct frequency components:

\[
E(z, t) = \sum_{n=1}^{N-1} |e_n \psi_n(z, t)\exp[i\nu_n(t - z/c)] + c.c.,
\]

where \(\nu_n\) denotes the (circular) carrier frequency of the \(n\)th component, \(e_n\) is its possible complex polarization vector, and \(\psi_n(z, t)\) is its complex amplitude, assumed to be a slowly varying function of \(z\) and \(t\) compared to the optical frequency. The frequencies \(\nu_n\) are chosen to be nearly resonant with the successive transition frequencies in a chain of \(N\)-dipole-connected energy levels in an atomic system, and \(C_n\) depends on the energy-level ordering, so that for increasing energies \(E_{n+1} > E_n, C_n = 1\), and for decreasing energies \(E_{n+1} < E_n, C_n = -1\). The use of \(C_n\) allows the Bloch equations or the density matrix equations for the evolution of the atomic variables in the rotating-wave approximation to have an invariant form for any energy-level ordering.

The evolution of the atomic system is described by the Liouville equation for the density matrix \(\varrho(t)\) of the system:

\[
\frac{d}{dt} \varrho(t) = \{H(t), \varrho(t)\},
\]

where for problems in which a rotating-wave approximation is used and in which the dipole transition moments link the levels only stepwise, 1-2, 2-3, \ldots, \((N - 1) - N\), the matrix elements of the Hamiltonian \(H(t)\) for a particular atom can be written as

\[
H_{nn}(t) = \hbar \Delta_n(t),
\]

\[
H_{n,n+1}(t) = H'_{n+1,n}(t) = -(1/2)\hbar \Omega_n(t),
\]

and

\[
H_{nn'}(t) = 0 \text{ otherwise}.
\]

Here \(\Delta_n(z, t)\), which generally depends on position and time, is the cumulative detuning of \(n - 1\) successive laser frequencies from the corresponding sum of \(n - 1\) Doppler-shifted transition frequencies, and \(\Omega_n(z, t)\) is the appropriate Rabi frequency.
\[ \Omega_n = 2\hbar^{-1}(d_{n,n+1} \cdot e_3) \delta_n(z, t), \]  
\[ n = 1, 2, \ldots, N \]  
where \( d_{n+1} = (n|d|n + 1) \) is the dipole moment between levels \( n \) and \( n + 1 \).

Equation (2), combined with the \( N - 1 \) reduced Maxwell equations given by
\[
\left[ \frac{\partial}{\partial t} + \frac{\partial}{\partial t(z)} \right] \Omega_n(z, t) = -\frac{4\pi D}{\hbar \epsilon_3} C_{n,n+1} d_n^2 (\rho_{n,n+1}),
\]
where \( d_n = |d_{n,n+1} \cdot e_3| \) is the absolute value of the appropriate component of the dipole matrix element, \( D \) is the atomic density, and \( (\cdot) \) denotes averaging over the Maxwellian velocity distribution of atoms, constitutes a semiclassical description of \( N - 1 \) electromagnetic pulses propagating in an atomic or molecular medium, with pulse lengths short compared with atomic or molecular relaxation times.

We shall now assume that the detunings \( \Delta_n(t) \) and Rabi frequencies \( \Omega_n(t) \) in Eqs. (3) are chosen to satisfy the following relations:
\[
\Delta_n(t) = -\{n - (1/2)(N + 1)\} \Omega_n(t), \quad n = 1, 2, \ldots, N
\]
and
\[
\Omega_n(t) = \sqrt{n(N - n)} \Omega_n(t), \quad n = 1, 2, \ldots, N - 1,
\]
where \( \Delta_0(t) \) and \( \Omega_0(t) \) can be arbitrary functions of time.

For the special case when \( \Delta_0(t) \) and \( \Omega_0(t) \) are independent of the time or have the same time dependence, Eqs. (6) are known as the Cook-Shore condition. More generally, an atomic or molecular system whose time-dependent Hamiltonian satisfies Eqs. (6) and (6) is said to possess the SU(2) model. Mathematically, the Hamiltonian of an N-level SU(2) model lies entirely in the subspace spanned by the generators of the \( O^+(3) \) subgroup of the \( N^2 - 1 \) generators of the SU(2) algebra, i.e., it can be written as
\[
\hat{H}(t) = c_1(t) \hat{J}_z + c_2(t) \hat{J}_y + c_3(t) \hat{J}_z + d(t), \tag{7}
\]
where \( \hat{J}_x, \hat{J}_y, \) and \( \hat{J}_z \) are the angular momentum operators of spin \( j = (1/2)N \). Since the presence of \( d(t) \) in Eq. (7) only changes the result for the probability amplitudes by the same phase factor, we shall ignore its presence. We use the representation in which \( \hat{J}_z \) is diagonal; Eq. (7) gives Eqs. (3) and (6) if we identify
\[
c_1(t) = i\hbar \Omega_0(t), \quad c_2(t) = -\hbar \Delta_0(t).
\]
If we use the labels \( m = -j, -j + 1, \ldots, j \) for the angular momentum states, which are related to the labels \( n = 1, 2, \ldots, N \) for an \( N \)-level system by
\[
m = n - j - 1, \quad N = 2j + 1,
\]then the matrix elements of the Hamiltonian are given by
\[
H_{nn}(t) = -\hbar \Delta_n(t), \tag{10a}
\]
\[
H_{nn+1}(t) = H_{nn-1,m}(t) = -\{1/2\} \{j - m\} \{j + m + 1\} \hbar \Omega_n(t), \tag{10b}
\]
\[
H_{nn}(t) = 0 \text{ otherwise.} \tag{10c}
\]
As was pointed out in Refs. 7 and 10, the special symmetry feature of the SU(2) model in the Liouville Eq. (2) is best revealed if we use the irreducible tensorial sets of Racah12-14 as the basis operators for the representation of \( \rho(t) \). In terms of the projection operator \( |m\rangle \langle m'| \), the \( N^2 \) operators of Racah, \( T_q^{(k)}(t) \), where for each \( k = 0, 1, 2, \ldots, j \), \( m \), \( m' \), are defined by
\[
T_q^{(k)} = \sum_{m,m'} \{(-1)^{(j-m')(2k+1)/2}(j \, k \, j)\}^{m'}_{m} |m\rangle \langle m'|
\]
where \( m, m' = -j, -j + 1, \ldots, j \) and where
\[
(j \, j \, j)_{m,m',m''} \]
is the Wigner 3-j symbol.11

An expression for the purpose of computing the 3-j symbols here is the following:
\[
(j \, j \, j)_{m,m',m''} = \delta(m_1 + m_2 + m_3, 0)(-1)^{j_1 - j_2 - m_1} \frac{1}{[(j_1 + j_2 + j_3)!]^2} \times \left[ \sum_{s} (-1)^{s} \frac{1}{s![(j_1 + j_2 - j_3 - s)!(j_1 - m_1 - s)!(j_2 + m_2 - s)!(j_3 - m_3)!][(j_3 - j_2 + m_1 + s)!(j_3 - j_2 + m_1 + s)!]} \right].
\]

Note the relationships
\[
T_q^{(k)}(t) = (-1)^{q} T_{-q}^{(k)}(t) \tag{12a}
\]
and
\[
\text{tr}[T_q^{(k)}(t) T_{-q'}^{(k')}(t)] = \delta(kk') \delta(qq'). \tag{12b}
\]

In terms of this basis set, the density matrix \( \rho(t) \) has \( N^2 \) components \( T_q^{(k)}(t) \), as expressed by
\[
\rho(t) = \sum_{k=0}^{2j} \sum_{q=-k}^{k} T_q^{(k)}(t) T_{q}^{(k)}, \tag{13a}
\]
where
\[
T_q^{(k)}(t) = \text{tr}[\rho(t) T_{q}^{(k)}(t)]. \tag{13b}
\]
Expressing the Liouville Eq. (2) in terms of the \( T_q^{(k)}(t) \), we find that
\[
i\hbar \frac{d}{dt} T_q^{(k)}(t) = \sum_{k',q'} A_{q'k'q}(t) T_{q'}^{(k')}(t), \tag{14a}
\]
where
\[
A_{q'k'q}(t) = -\text{tr}[\hat{H}(t) T_{q'}^{(k')}(t)]. \tag{14b}
\]
At this point, the special feature of the SU(2) model, as expressed by its Hamiltonian given by Eqs. (10), will be noticed, for it can be shown\(^\text{10}\) from Eq. (14b) that \(A_{q,kq'}(t) = 0\) unless \(k = k'\). That is to say, the \(N\)-dimensional space in which the atomic variables evolve decomposes, in the basis set of Racah, into \(N\) independent subspaces of dimensions \(2k + 1, \ldots , 2j\), i.e., of dimensions 1, 3, 5, \ldots , \(4j + 1\). In each subspace characterized by the value of \(k\), the equation of motion for \(T_q^{(k)}(t)\) is given by\(^\text{15}\)

\[
\frac{ih}{d}T_q^{(k)}(t) = \hat{A}(t)T_q^{(k)}(t),
\]

(15)

where the matrix elements of the matrix \(\hat{A}\) are given by

\[
A_{q,-k}(t) = -q\hbar\Delta_0(t), \quad (16a)
\]

\[
A_{q+1,-q}(t) = A_{q+1}(t) = -(1/2)[(k - q)(k + q + 1)]^{1/2}\hbar\Delta_0(t), \quad (16b)
\]

\[
A_{q,-k}(t) = 0 \text{ otherwise,} \quad (16c)
\]

where \(k = 0, 1, 2, \ldots , j\) and \(q = -k, -k + 1, \ldots , k\).

Notice that Eqs. (16) exactly parallel Eqs. (10) except that \(k\) takes only integer values here while \(j\) in Eqs. (10) takes half-integer as well as integer values.

Following the solution given by me in Ref. 9, the solution of Eq. (15) can be readily written down. Let \(a(t)\) and \(-b^*(t)\) denote the solutions for \(C_1(t)\) and \(C_2(t)\), respectively, satisfying the following equations of motion:

\[
\frac{id}{dt} \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} = \begin{bmatrix} (1/2)\Delta_0(t) & -(1/2)\Delta_0(t) \\ -(1/2)\Delta_0(t) & -(1/2)\Delta_0(t) \end{bmatrix} \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix},
\]

(17)

with the initial conditions \(C_1(0) = 1, C_2(0) = 0\), so that the solution of Eqs. (17) for the general initial values of \(C_1(0)\) and \(C_2(0)\) can be written as

\[
\begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} = \begin{bmatrix} a(t) & b(t) \\ -b^*(t) & a^*(t) \end{bmatrix} \begin{bmatrix} C_1(0) \\ C_2(0) \end{bmatrix}.
\]

(18)

Then for each value of \(k\) in Eq. (15) the solution for \(T_q^{(k)}(t)\) is given by

\[
T_q^{(k)}(t) = \sum_{q = -k}^{k} D_{qq'}(a, b)T_{q'}^{(k)}(0),
\]

(19)

where \(D_{qq'}(a, b)\) are the matrix elements of the \((2k + 1)\)-dimensional representation of the \(SU(2)\) group.\(^\text{16}\) The following expression:

\[
D_{mm'}(a, b) = \sum_k \frac{[j(j + m)(j + m')]}{pjqb^*q!} \times a^p a^q b^q (b^* b)^m,
\]

(20)

If \(\rho_{mn}(0) = 0\) for \(m \neq n'\), \(\rho_{mn}(0)\) is an arbitrary real constant that may be positive or negative subject only to the condition that it should make

\[
\rho_{mn}(0) \geq 0 \quad \text{for all} \quad m = -j, -j + 1, \ldots , j.
\]

(25)

This can be shown as follows. From Eq. (13b) we find that

\[
T_q^{(k)}(t) = \sum_m \frac{(-1)^{j+m+q} j^m(2k + 1)^{1/2}}{q!} \times \left( \begin{array}{c} j \\ m \end{array} \right) \rho_{qm}(0),
\]

(26)

where \(T_q^{(k)}(0) = 0\) for \(q \neq 0\), then substituting Eq. (24) into Eq. (26) gives, for \(k \neq 0\),

\[
T_q^{(k)}(0) = 0 \quad \text{for} \quad q \neq 0
\]

(27a)

and

\[
T_0^{(k)}(0) = T_0^{(k)}(0)\delta(k, k_1),
\]

(27b)

where we have made use of the orthogonality property of the \(3-j\) symbols\(^\text{11}\):
The N - 1 sets of initial-level population given by setting 
k_1 = 1, 2, \ldots, N - 1 successively in Eq. (24) are independent
and orthogonal in the sense that any given arbitrary dis-
tribution of initial-level population p(0) can be expressed as a
superposition of these sets:
\[ \rho_{mm}(0) = (2j + 1)\delta_{0,0} + \sum_{k=1}^{2j} (-1)^{-m}(2k + 1)^{1/2} \times \left( \begin{array}{ccc} j & k & j \\ -m - q & m & m \end{array} \right) T^{(k)}(0) |m\rangle \langle m|, \]
where
\[ T^{(k)}(0) = \sum_{m} \rho_{mm}(0)(-1)^{-m}(2k + 1)^{1/2} \left( \begin{array}{ccc} j & k & j \\ -m - q & m & m \end{array} \right) T^{(k)}(t). \]

The special choice of the initial-level population as given
by Eq. (24) in which we set k_1 = 1, 2, \ldots, N - 1 successively
greatly simplifies our expressions for the atomic variables
for each case. For each value of k_1, we obtain T^{(k)}(t) from
Eq. (22) and then obtain \rho_{m+q,m}(t) and \rho_{m,m+q}(t) from
\rho_{m+q,m}(t) = (2j + 1)\delta_{0,0} + (-1)^{-m}(2k + 1)^{1/2} \times \left( \begin{array}{ccc} j & k & j \\ -m - q & m & m \end{array} \right) T^{(k)}(t)
and
\rho_{m,m+q}(t) = (2j + 1)\delta_{0,0} + (-1)^{-m}(2k + 1)^{1/2} \times \left( \begin{array}{ccc} j & k & j \\ -m - q & m & m \end{array} \right) T^{(-k)}(t). \]

The atomic variables that we require for the reduced Max-
well Eqs. (5) are \rho_{m,m+1}(t), given by
\rho_{m,m+1}(t) = (-i)(-1)^{-m}(2k + 1)^{1/2} \times \left( \begin{array}{ccc} j & k & j \\ -m - 1 & m & m \end{array} \right) T^{(-k)}(t).
Since the \Omega_m(z, t) in Eq. (5) are already chosen to be given by
\Omega_m(z, t) = [(j - m)(j + m + 1)]^{1/2} \Omega(z, t),
the set of Eqs. (5) with \rho_{m,m+1}(t) given by Eqs. (33) substi-
tuted into them would give a consistent result if the following
relations are satisfied:
\[ \frac{C_m C_{m+1}^2}{[(j - m)(j + m + 1)]^{1/2}} (-1)^{-m} \left( \begin{array}{ccc} j & k & j \\ -m - 1 & m & m \end{array} \right) = \text{const.} \]
for m = -j, -j + 1, \ldots, j - 1. When Eqs. (35) are satisfied,
\[
\frac{1}{\tau^2} = \frac{2\pi D}{\hbar c} C_n d_m^2 \rho_{mm}(0) - \rho_{m+1,m+1}(0),
\]  
(45)

and where \( f^{(h)}(a, b) \) is given by Eq. (40). The specific examples that we shall give below will show that a solitary-pulse solution for Eq. (44) requires that \( 1/\tau^2 \) be positive.

Equations (24) and (35) constitute the two conditions that must be satisfied so that the equation for the propagation of the common pulse amplitude specified by \( \Omega_0(z, t) \) is determined by Eq. (44), which, as we shall see, permits special solitary-pulse solutions.

More specifically, the necessary conditions for the initial-level population as well as for the frequencies \( (\rho_m) \) of the lasers, the dipole moment matrix elements \( (d_m) \), and the level configuration \( (C_m) \), specified by Eqs. (24) and (37), are, for

\[
k_1 = 1, \quad \rho_{mm}(0) = m\delta + (2j + 1)^{-1}, \quad C_n d_m^2 = \text{const.,}
\]  
(46a)

\[
k_1 = 2, \quad \rho_{mm}(0) = [3m^2 - j(j + 1)]\delta + (2j + 1)^{-1}, \quad C_n d_m^2 (2m + 1) = \text{const.,}
\]  
(46b)

\[
k_1 = 3, \quad \rho_{mm}(0) = [5m^2 - (3j^2 + 3j - 1)m]\delta + (2j + 1)^{-1}, \quad C_n d_m^2 [2 + 5m(m + 1) - j(j + 1)] = \text{const.,}
\]  
(46c)

\[
k_1 = 4, \quad \rho_{mm}(0) = [70m^4 - 60j(j + 1) - 50]m^2 + 6(j - 1)(j + 1)(j + 2)\delta + (2j + 1)^{-1}, \quad C_n d_m^2 [14m^2 + 21m + 19 - 6j(j + 1)]m + [6 - 3j(j + 1)] = \text{const.,}
\]  
(46d)

where \( \delta \) can be positive or negative subject only to the requirement that \( \rho_{mm}(0) \geq 0 \) for all \( m = -j, -j + 1, \ldots, j \) and where \( j = (1/2)(N - 1) \).

For example, for a five-level \((N = 5 \text{ or } j = 2)\) system whose Hamiltonian satisfies Eqs. (5) and (6) to support the simultaneous propagation, the conditions expressed by Eqs. (24) and (37), as well as the necessary level configurations or ordering required by \( 1/\tau^2 \) being positive in Eq. (45), for the four mutually exclusive cases characterized by \( k_1 = 1, 2, 3, 4 \) are given explicitly in Table 1, where \( \rho_{ji}(0) \) denotes \( \rho_{ji}(0) - (1/5) \). The corresponding level configurations are sketched in Figs. 1(a), 1(b), 1(c), and 1(d), respectively.

The quantity \( f^{(h)}(a, b) \) given by Eq. (40) can be expressed in terms of the Jacobi polynomial \( P_n(\alpha, \beta)(x) \) as follows:
Fig. 1. (a), (b), (c), (d) Energy-level configurations for the five-level system for the cases $k_1$ equal to 1, 2, 3, and 4, respectively, given in Table 1.

\[ b(z, t) = i \sin(1/2) \theta, \quad \text{(54b)} \]

so that

\[ -2iab = \sin \theta, \quad \text{(55)} \]

\[ |a|^2 - |b|^2 = \cos \theta, \quad \text{(56)} \]

\[ f^{(k_1)}(\theta) = \frac{-2}{k_1(k_1 + 1)} \frac{d}{d\theta} P_{k_1}(\cos \theta), \quad \text{(57)} \]

and Eq. (53) becomes

\[ \tilde{\phi} = \frac{-2}{r^2 k_1(k_1 + 1)} \frac{d}{d\theta} P_{k_1}(\cos \theta). \quad \text{(58)} \]

The two angles, $\theta$ and $\tilde{\phi}$, given by Eqs. (54) and (43), respectively, are identical, as can be seen from the solution of the two-level system, Eqs. (17), when $\omega_0(z, t)$ is real and $\Delta_0(z, t) = 0$. Thus Eq. (58) becomes

\[ \tilde{\phi} = \frac{1}{r^2} f^{(k_1)}(\theta), \quad \text{(59)} \]

where from Eq. (57) we find that \(^{21}\)

\[ f^{(k_1)}(\theta) = \begin{cases} \sum_{r=0}^{n} c_{2n-2r+1} \sin(2n - 2r + 1) \theta & \text{for } k_1 = 2n + 1 \\ \sum_{r=0}^{n-1} c_{2n-2r} \sin(2n - 2r) \theta & \text{for } k_1 = 2n \end{cases} \quad \text{(60a)} \]

and

\[ c_{2n-2r} = \frac{1}{k_1(k_1 + 1)} \frac{1}{2^{4n-3} (r!)^2[(2n + 1 - r)!]^2} (2n - 2r + 1) \quad \text{(61a)} \]

\[ f^{(k_1)}(\theta) = \begin{cases} 1 & \text{for } k_1 = 2n + 1 \\ (1/2) \sin 2\theta & \text{for } k_1 = 2n \end{cases} \quad \text{(60b)} \]

Equations (60) and (61) show that $f^{(k)}(\theta)$ is a sum of $\sin k\theta$, $\sin(k - 2)\theta$, $\ldots$, with positive coefficients. For example, for $k = 1 - 4$, $f^{(k)}(\theta)$ are given by

\[ f^{(1)}(\theta) = \sin \theta, \quad \text{(63a)} \]

\[ f^{(2)}(\theta) = (1/2) \sin 2\theta, \quad \text{(63b)} \]

\[ f^{(3)}(\theta) = (1/16) (\sin \theta + 5 \sin 3\theta), \quad \text{(63c)} \]

\[ f^{(4)}(\theta) = (1/32) (2 \sin 2\theta + 7 \sin 4\theta). \quad \text{(63d)} \]

Equation (59) can be readily integrated to give
\[ \frac{d^2}{dx^2} \left[ 1 - P_{k_1} \cos \theta \right] = \frac{4}{r^2 k_1 (k_1 + 1)} \]  

(64)

where we have assumed that \( \theta = \dot{\theta} = 0 \) at the initial time.

When the electric-field envelope area is defined as in Eq. (43),

\[ A(z, t) = \int_0^t \Omega_0(z, t') dt', \]

(65)

the corresponding area theorem of McCall and Hahn\(^1\) is

\[ \frac{\partial}{\partial z} A(z, t) = -\frac{1}{2} \alpha f^{(k_1)}(A), \]

(66)

where \( \alpha \) is the absorption coefficient and where \( f^{(k_1)}(A) \) is given as in Eqs. (57) and (60). Thus, if \( A = n \pi \) for any positive integer \( n \), the pulse envelope area suffers no attenuation in propagation since \( \partial A / \partial z = 0 \). The areas that are even multiples of \( \pi \) are known to be more stable than those that are odd multiples.\(^3\) These pulses travel anomalously slowly but behave as if the medium were transparent in the sense that they suffer no attenuation during propagation.

Equation (64) shows that the simulton propagation problem can be reduced to quadratures, given the conditions for any \( k_1 \). If we set \( x = \cos \theta \) in Eq. (64), the integrated solution is given by

\[ \int_{\xi}^{x} \frac{dx}{[1 - x^2][1 - P_{k_1}(x)]^{1/2}} = -\frac{2}{r [k_1 (k_1 + 1)]^{1/2}} (\xi - \xi_0). \]

The integral on the left-hand side can be expressed in terms of elementary functions or elliptic integrals for the cases \( k_1 = 1 \) and \( k_1 = 6 \). The explicit analytic expressions for \( \theta \) in terms of \( \xi - \xi_0 \) are simple for the first two cases \((k_1 = 1, 2)\). For \( k_1 = 1 \), Eq. (59) becomes

\[ \dot{\theta} = \frac{1}{r^2} \sin \theta, \]

(67)

and the solitary-pulse solution is

\[ \dot{\theta} = \Omega_0(\xi) = \frac{2}{\tau} \text{sech} \left( \frac{\xi - \xi_0}{\tau} \right), \]

(68a)

or

\[ \theta = 4 \tan^{-1} \left( \exp \left( \frac{\xi - \xi_0}{\tau} \right) \right). \]

(68b)

We have assumed that Eqs. (46a) are satisfied. In addition, the requirement that \( 1/r^2 \) be positive in Eq. (45) requires that \( C_m(\rho_{m-1} - \rho_{m+1}) \) be positive, and this can be satisfied if the levels are of the cascade configuration, i.e., \( E_{n+1} > E_n \) for all \( n \) or \( E_{n+1} < E_n \) for all \( n \), with the level population arranged accordingly [Eqs. (46a)]. The simultons consisting of \( N - 1 \) solitary pulses possibly of different frequencies but of the same speed \( V \) are given, from Eqs. (6), by

\[ \Omega_n(\xi) = \sqrt{n(N - n)} \Omega_0(\xi), \quad n = 1, 2, \ldots, N - 1. \]

(69)

This is the simulton solution given in Refs. 1 and 2. For \( k_1 = 2 \), Eq. (59) becomes

![Fig. 2. Solitary-pulse solutions \( \dot{\theta} \) of Eq. (59) or (64) as a function of \( \xi \) (setting \( \xi_0 = 0 \) and \( \tau = 1 \)) for the cases \( k = k_1 = 1-6 \). The area under each pulse is equal to \( 2\pi \) for \( k \) odd and to \( \pi \) for \( k \) even.](image)
\[ 2\bar{\theta} = \frac{1}{\tau^2} \sin 2\theta, \]  
\( (70) \)

and the solitary-pulse solution is
\[ \dot{\theta} = \Omega_0(t) = \frac{1}{\tau} \sech \frac{t - \bar{\theta}_0}{\tau}, \]  
\( (71a) \)
or
\[ \theta = 2 \tan^{-1} \left( \exp \frac{t - \bar{\theta}_0}{\tau} \right). \]  
\( (71b) \)

We have assumed in this case that Eqs. (46b) are satisfied. In addition, the requirement that \( 1/\tau^2 \) be positive in Eq. (45) requires that the number of levels be odd, with the energies \( E_{n+1} > E_n \) (or \( E_{n+1} < E_n \)) for \( n = 1, 2, \ldots, (N - 1)/2 \), and \( E_n < E_{n+1} \) (or \( E_n > E_{n+1} \)) for \( n = (N + 1)/2, (N + 1)/2 + 1, \ldots, N \). This is the simulton solution given in Ref. 6.

For \( k_1 > 2 \), the solitary-pulse solution of Eq. (59) or (64) cannot be expressed simply. Instead, we shall examine the shapes of these solitary pulses numerically computed from Eq. (64) and point out their general features. They are presented in Fig. 2 for \( k_1 = 3-6 \) together with the familiar hyperbolic-secant pulse for \( k_1 = 1, 2 \). It will be noted that the area of the pulse given by
\[ A = \lim_{\tau \to +\infty} \theta(t) \]
is
\[ A = \begin{cases} 2\pi & \text{for } k_1 \text{ odd} \\ \pi & \text{for } k_1 \text{ even} \end{cases}, \]  
\( (72) \)

which can be understood from Eq. (64) by noting that \( P_n(-1) = (-1)^n \) and \( P_n(1) = 1 \), and hence \( \dot{\theta} \), starting from zero, approaches its next zero as \( \theta \) approaches \( \pi \) and \( \pi \) for \( k_1 \) odd and \( k_1 \) even, respectively. The solitary pulse has \( k_1 \) maxima and \( k_1 - 1 \) minima if \( k_1 \) is odd and has \( k_1/2 \) maxima and \( k_1/2 - 1 \) minima if \( k_1 \) is even. In terms of \( \theta(t) \), which is given by
\[ \theta(t) = \int_{-\infty}^{t} \dot{\theta}(t) dt, \]  
\( (73) \)
the maxima or minima of \( \dot{\theta}(t) \) as a function of \( t \) occur at the values \( \theta_m \) given by
\[ \frac{d}{d\theta} P_{k_1}(\cos \theta) = 0. \]  
\( (74) \)

Since the pulse is symmetrical about \( t = t_0 \) (which may be taken to be zero), for \( k_1 \) odd, \( t = t_0 \) corresponds to \( \theta = \pi \), and it always gives a maximum of \( \dot{\theta} \); but for \( k_1 \) even, \( t = t_0 \) corresponds to \( \theta = \pi/2 \), and it gives a maximum of \( \dot{\theta} \) if \( k_1/2 \) is an odd integer and a minimum of \( \dot{\theta} \) if \( k_1/2 \) is an even integer. The value of \( \dot{\theta} \) at the maximum or minimum corresponding to \( \theta = \theta_m \) given by Eq. (74) is
\[ \dot{\theta}_{\theta = \theta_m} = \frac{4}{\tau^2 P_{k_1}(1)} \left[ 1 - P_{k_1}(\cos \theta_m) \right]^{1/2}. \]  
\( (75) \)

It is useful to note that
\[ P_n(0) = \begin{cases} 0 & \text{for } n \text{ odd} \\ \left( -1 \right)^{(1/2)n}n! & \text{for } n \text{ even} \end{cases}, \]  
\( (76) \)

\[ \text{At } t \to \pm\infty, \dot{\theta}(t) \text{ approaches zero exponentially. Since the pulse area } A = \pi n, \text{ it is seen from Eqs. (60) and (66) that the pulse envelope area will suffer no attenuation in propagation.} \]

To summarize our results, we began with an atomic medium consisting of identical atoms each of which had generally \( N \) transition levels that were chainwise dipole connected, and we considered sending \( N - 1 \) simultaneous equal-velocity laser pulses of possibly different wavelengths through the medium. We assumed that the time-dependent Hamiltonian of the laser–atom interacting system satisfied Eqs. (10). The system was said to possess the SU(2) dynamic symmetry. We assumed that Eqs. (24) and (37) were satisfied. That means that, given an atomic medium for which the \( N - 1 \) dipole moments \( \delta m \) of the \( N \) transition levels of each atom were given, the level configuration \( C_m \) was assumed to be one of the \( N - 1 \) possibilities (see the example for \( N = 5 \) given in Fig. 1), and the laser frequencies \( \nu_p \) and the initial level population \( n_{m0} \) (0) were assumed to have been chosen appropriately so that Eqs. (24) and (37) were satisfied, the order \( k_1 \) of the possible simulton solution being determined by the level configuration of the atoms. When the above conditions are satisfied, simultons (of order \( k_1 \)) consisting of \( N - 1 \) \( \Omega_n(t) \) given by Eq. (69), where \( \Omega_n(t) \) or \( \dot{\theta} \) is given by the solution of Eq. (59) or (64), can propagate through the atomic medium without attenuation.

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**Note Added in Proof:** It was pointed by a referee that the integrated solitary-pulse solution of Eq. (59) or (64) for the case \( k_1 = 4 \) can be expressed in a rather simple and compact form. It is
\[ \dot{\theta} = \frac{\sqrt{2}}{\tau} \cosh \left[ -\left( t - \bar{\theta}_0 \right)/\tau \right] \]  
or
\[ \theta = \tan^{-1} \sqrt{2} \cosh \left[ -\left( t - \bar{\theta}_0 \right)/\tau \right]. \]

The two maxima of \( \dot{\theta} \) occur at \( (t - \bar{\theta}_0)/\tau = \pm \ln \sqrt{2} \) = ±1.628307.

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20. See Ref. 19, pp. 782 and 344.


22. See Ref. 19, p. 776.