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Abstract

We present the results of applying an analytical method, using a new concept involving a periodic stability around a line of initial values, to two Hamiltonian systems with two degrees of freedom each with several parameters. By requiring the systems to have this type of stability, we have been led to known integrable cases for the systems. For one of the systems, our analysis gives six other cases, two of which turn out to be nonintegrable. The status of the remaining four cases has not been established.

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Certain stability type and integrable two-dimensional Hamiltonian systems

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We present the results of applying an analytical method, using a new concept involving a periodic stability around a line of initial values, to two Hamiltonian systems with two degrees of freedom each with several parameters. By requiring the systems to have this type of stability, we have been led to known integrable cases for the systems. For one of the systems, our analysis gives six other cases, two of which turn out to be nonintegrable. The status of the remaining four cases has not been established.

I. INTRODUCTION

The problem of integrability and local and global stability and instability of a dynamical system has been a subject of intense studies for many years.[1] For a Hamiltonian system with N degrees of freedom, integrability means the existence of N independent analytic global integrals of the motion. There have been several ingenious discoveries of integrable Hamiltonian systems through applications of inverse scattering transforms.[2] Of great importance and interest also is the suggestion that dynamical systems with the so-called Painlevé property are integrable.[3-9]

We confine ourselves to Hamiltonian systems whose Hamiltonians are assumed to contain several parameters $A,B,C,D,\ldots$. The local stability or instability of a class of periodic motions as a function of one of the parameters, C, say, that couples the motion in different spatial dimensions $x,y,z,\ldots$ was previously studied by Deng and the author.[10,11] It was found that generally the motion would undergo many or even an infinite number of stability-instability transitions[10-17] as C is varied from $-\infty$ to $+\infty$; we exclude the region where the motion may become unbounded. The transition points depend on the initial values of the $2N$ variables $x,\dot{x},y,\dot{y},z,\dot{z},\ldots$, but it was found that the exponent by which the Lyapunov exponent approaches zero as C approaches any one of its critical values from the unstable region is universal for Hamiltonian systems of any dimension for the class of periodic motions described.[11]

Local stability does not, of course, imply global integrability of the system. In this paper, however, we shall expand the consideration of stability for a given initial phase space point to that for a line of initial values in this line, and if any small deviations from this set of initial values obey a certain behavior to be specified in the following section, we shall call the stability that of type 1 for brevity, where the 1 is meant to refer to a one-dimensional line of initial values, plus a specific condition to be attached to the behavior of any small deviations to be explained in Sec. II.

We consider the generalized models of two well-known two-dimensional Hamiltonian systems: the coupled quartic oscillators[18] and the Hénon-Heiles[19] system. The generalized models contain five and four parameters $A,B,C,\ldots$, respectively. Our analytical analysis of the stability of type 1 for the systems has led to the following results: (1) For the generalized coupled quartic oscillator system, we have recovered all the previously known integrable cases when we required the system to have stability of type 1 with respect to two specific lines of initial values; (2) for the generalized Hénon-Heiles system, we have been able to recover all but one of the previously known integrable cases when we required the system to have stability of type 1 with respect to a specific line of initial values. In addition, we have found six other cases which have stability of type 1 with respect to the same line of initial values for (2). Two of these six cases turn out to be nonintegrable (see *Notes added*), and the status of the remaining four cases has not been established.

We should mention that Yoshida,[20] Ito,[21] and Yoshida *et al.*[22] have made use of the straight-line periodic solution in conjunction with Ziglin's theorem[23] to show the conditions for the nonintegrability of a system. Our analysis in this paper complements their analysis in some way in that we have introduced a new concept of stability of type 1 to suggest a possible close relation between systems which have such property and integrable systems.

II. STABILITY OF TYPE 1

Consider a general two-dimensional Hamiltonian system whose Hamiltonian is

$$H=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)+V(x,y)\ , \tag{2.1}$$

where the potential energy $V(x,y)$ may contain several parameters $A,B,C,D,\ldots$. Specifically, the two Hamiltonian systems which we shall consider have the following potential energies:

$$V(x,y)=\begin{cases}\tfrac{1}{2}(Ax^2+By^2)+Dx^4+2Cx^2y^2+Ey^4 & \text{(2.2a)}\\ \tfrac{1}{2}(Ax^2+By^2)+Cx^2y-\dfrac{D}{3}y^3\ , & \text{(2.2b)}\end{cases}$$

where the systems with (2.2a) and (2.2b) will be referred to as the generalized coupled quartic oscillator system and the generalized Hénon-Heiles system, respectively. The parameter C has been used in both cases for the coefficient of the coupling term in x and y, and the parameters A and B are related to the frequencies of the "harmonic" terms. The motion of the systems, or of a "particle" in these potential wells, is completely determined once the initial values $x(0)$, $y(0)$, $\dot{x}(0)$, $\dot{y}(0)$ are given. It may be regular or chaotic, depending on the initial values and on the shape of the potential wells determined by the parameters $A,B,C,\ldots$; we consider only the bounded motions.

We now consider a special set of initial values such that the motion is a simple periodic motion in one dimension with a specific frequency, for all allowed values of the parameters $A,B,C,\ldots$. A typical set of such initial values, as we shall see in the following sections, is $x(0)=\dot{x}(0)=\dot{y}(0)=0$, and $y(0)=y_0$ where y_0 can take on a range of values that do not make the motion unbounded. Next, by keeping, say, $\dot{x}(0)=\dot{y}(0)=0$, but letting $x(0)=\Delta x$ and $y(0)=y_0+\Delta y$, where Δx and Δy are small perturbations, the motion may remain stable or become unstable depending on y_0 and on the parameters $A,B,C,\ldots$. We say that the system is stable with stability of type 1 with respect to the line of initial values $(x(0),y(0),\dot{x}(0),\dot{y}(0))=(0,y_0,0,0)$ if Δx and Δy not only remain bounded but are periodic functions of the time, whose periods may depend on y_0, for all allowable values of y_0. Mathematically, this requires Δx and Δy to be the characteristic functions with well-defined periods of the appropriate linearized differential equations (with generally periodic coefficients) for Δx and Δy for all allowable values of y_0, where the periods may depend on y_0.

The requirement that the system be stable with stability of type 1 for some chosen line or lines of initial values imposes conditions on the values of the parameters $A,B,C,D,\ldots$ in the Hamiltonian. We shall see what this means for Hamiltonian systems whose potential energies are given by (2.2a) and (2.2b), respectively, in the following sections.

III. GENERALIZED COUPLED QUARTIC OSCILLATOR SYSTEM

For the generalized coupled quartic oscillator system whose potential energy in the Hamiltonian (2.1) is given by (2.2a), the equations of motion are given by

$$\ddot{x}=-Ax-4(Dx^3+Cxy^2)\ , \tag{3.1a}$$

$$\ddot{y}=-By-4(Ey^3+Cx^2y)\ . \tag{3.1b}$$

For small deviations Δx and Δy, their behavior is governed by the following equations:

$$\frac{d^2(\Delta x)}{dt^2}+(A+12Dx^2+4Cy^2)(\Delta x)+8Cxy(\Delta y)=0\ , \tag{3.2a}$$

$$\frac{d^2(\Delta y)}{dt^2}+(B+12Ey^2+4Cx^2)(\Delta y)+8Cxy(\Delta x)=0\ . \tag{3.2b}$$

If $y(0)=\dot{y}(0)=0$, then $y(t)=0$ for all t, as can be seen from Eq. (3.1b), and the equation of motion for x is, from Eq. (3.1a),

$$\ddot{x}+Ax+4Dx^3=0\ , \tag{3.3}$$

for which the solution for $x(0)=x_0$, $\dot{x}(0)=0$ is

$$x=x_0\mathrm{cn}(\tau,k)\ , \tag{3.4}$$

where $\mathrm{cn}(\tau,k)$ is the Jacobi elliptic function of modulus k and where

$$\tau=\Omega t\ , \tag{3.5}$$

$$\Omega=(A+4Dx_0^2)^{1/2}\ , \tag{3.6}$$

and

$$k^2=\frac{2Dx_0^2}{A+4Dx_0^2}\ . \tag{3.7}$$

Thus x is a periodic function whose period in the rescaled time τ is $4K(k)$, where $K(k)$ is the complete elliptic integral of the first kind. Substituting Eqs. (3.4)–(3.7) into (3.2), we find that the behavior of any small deviations Δx and Δy from the chosen line of initial values given by $(x(0),y(0),\dot{x}(0),\dot{y}(0))=(x_0,0,0,0)$ is governed by the following equations:

$$\frac{d^2(\Delta x)}{d\tau^2}=[6k^2\mathrm{sn}^2(\tau)-(1+4k^2)](\Delta x)\ , \tag{3.8a}$$

$$\frac{d^2(\Delta y)}{d\tau^2}=\left[\frac{2C}{D}k^2\mathrm{sn}^2(\tau)-\left[\frac{2C}{D}k^2+\frac{B}{A}(1-2k^2)\right]\right](\Delta y)\ . \tag{3.8b}$$

Equations (3.8) are of the form of the Lamé equation[24]

$$\frac{d^2u}{d\tau^2}=[n(n+1)k^2\mathrm{sn}^2(\tau)-h]u\ , \tag{3.9}$$

where $\mathrm{sn}(\tau)$ has the modulus k and where n may be referred to as the order of the equation which need not be an integer. Periodic solutions of the Lamé equation, called the periodic Lamé functions, exist for certain characteristic values of h. In particular it was shown by Ince[24] that periodic Lamé functions of real periods $2K$, $4K$, and $8K$ corresponding to integral and half integral values of n have characteristic values which can be expressed in simple analytic forms.

Requiring the system to be stable with stability of type 1 for the given line of initial values implies that the solutions of Δx and Δy from Eqs. (3.8) must be periodic Lamé functions for all allowed values of k. An examination of Eqs. (3.8) shows that we need periodic Lamé functions whose characteristic values are of the form $a+bk^2$, where a and b are real constants. In the Appendix, we have listed all the periodic Lamé functions that we can determine whose characteristic values are of the form

$$a+bk^2+c(1-k^2+k^4)^{1/2}\ , \tag{3.10}$$

where we have included the third term in (3.10) for the case of the generalized Hénon-Heiles system to be discussed in the following section. From Eq. (3.8a) and from the Appendix, we see that for any value of x_0 or $0 \le k^2 \le \frac{1}{2}$, Δx is given by the Lamé periodic polynomial $\mathrm{Es}_2^1(\tau)$ with the characteristic value equal to $b_2^1 = 1+4k^2$, for any values of $A,B,\ldots,E$ in the Hamiltonian. On the other hand, for Δy in Eq. (3.8b) to be periodic and thus to be a periodic Lamé function, we must first match $(2C/D)k^2+(B/A)(1-2k^2)$ with the characteristic values given in the Appendix, and then with the value of C/D so required, we must check whether it agrees with the order n of the characteristic value and function given by $n(n+1)=2C/D$. When these are satisfied, then Δy will be given by the periodic Lamé function and the system has stability of type 1 with respect to the chosen line of initial values.

The result of the above analysis gives the following four possibilities for the system to have stability of type 1: (1)

$$B/A = \tfrac{1}{4}, \quad C/D = \tfrac{3}{8} , \tag{3.11}$$

for which Δy is equal to some linear combinations of $\mathrm{Ec}_{1/2}^{1/2}(\tau)$ and $\mathrm{Es}_{1/2}^{1/2}(\tau)$, and for which the characteristic value is $a_{1/2}^{1/2}$; (2)

$$B/A = 1, \quad C/D = 1 , \tag{3.12}$$

for which $\Delta y = \mathrm{Ec}_1^1(\tau)$, and the characteristic value is a_1^1; (3)

$$B/A = 1, \quad C/D = 3 , \tag{3.13}$$

for which $\Delta y = \mathrm{Es}_2^1(\tau)$, and the characteristic value is b_2^1; and (4)

$$B/A = 4, \quad C/D = 6 , \tag{3.14}$$

for which $\Delta y = \mathrm{Es}_3^2(\tau)$, and the characteristic value is b_3^2.

It is seen that the value of E in the Hamiltonian can be arbitrary for the system to have stability of type 1 with respect to the line of initial values $(x(0),y(0),\dot{x}(0),\dot{y}(0))=(x_0,0,0,0)$.

If we carry out the similar analysis for the line of initial values $(x(0),y(0),\dot{x}(0),\dot{y}(0))=(0,y_0,0,0)$, it is easy to see that for the system to have stability of type 1 with respect to this line, we would obtain the following four possibilities: (1′)

$$A/B = \tfrac{1}{4}, \quad C/E = \tfrac{3}{8} , \tag{3.15}$$

(2′)

$$A/B = 1, \quad C/E = 1 , \tag{3.16}$$

(3′)

$$A/B = 1, \quad C/E = 3 , \tag{3.17}$$

(4′)

$$A/B = 4, \quad C/E = 6 , \tag{3.18}$$

while the value of D in this case can be arbitrary.

If we now require the system to have stability of type 1 for both lines of initial values $(x_0,0,0,0)$ and $(0,y_0,0,0)$, then combining possibilities (2) and (2′), (3), and (3′), (4), and (1′), and (1) and (4′), respectively, give the first four possibilities listed in Table I.

Cases I and II in Table I are well-known integrable cases.[4,18] This can be readily proved as the Hamiltonians are separable in the two respective cases in the polar coordinates and in the x'-y' coordinates obtained from the x-y coordinates by a 45° rotation. Case III (or case IV by the obvious symmetry of the Hamiltonian) is also integrable.[7,9] It is separable in the parabolic coordinates $x=\xi\eta$, $y=\frac{1}{2}(\xi^2-\eta^2)$. The Hamilton's characteristic function in the corresponding Hamilton-Jacobi equation[25] can be written as the sum of two functions, one of ξ only and one of η only. The second integral of motion is, for case III,

$$\dot{x}(x\dot{y}-\dot{x}y)+x^2y[A+4D(x^2+2y^2)]=\text{const} . \tag{3.19}$$

By carefully examining Eqs. (3.11)–(3.18) under a certain limiting condition, we obtain two other possibilities for the system to have stability of type 1. First we notice from Eq. (3.7) that $k^2 \to \frac{1}{2}$ for the line of initial values $(x_0,0,0,0)$ for all values of x_0 if $A \to 0$. Similarly, for the line of initial values $(0,y_0,0,0)$, $k^2 \to \frac{1}{2}$ for all values of y_0 if $B \to 0$. Consider Eqs. (3.8), together with the equations corresponding to them for any small deviations Δx and Δy from the line of initial values $(x(0),y(0),\dot{x}(0),\dot{y}(0))=(0,y_0,0,0)$ given by

$$\frac{d^2(\Delta x)}{d\tau^2} = \left[\frac{2C}{E}k^2\mathrm{sn}^2(\tau) - \left[\frac{2C}{E}k^2 + \frac{A}{B}(1-2k^2) \right] \right](\Delta x), \tag{3.20}$$

$$\frac{d^2(\Delta y)}{d\tau^2} = [6k^2\mathrm{sn}^2(\tau)-(1+4k^2)](\Delta y) , \tag{3.21}$$

where $k^2=2Ey_0^2/(B+4Ey_0^2)$. If we set $B/A = A/B = 1$, $C/D=3$, and $C/E=\frac{3}{8}$ in Eqs. (3.8b) and (3.20), then these equations reduce respectively to

$$\frac{d^2(\Delta y)}{d\tau^2} = [6k^2\mathrm{sn}^2(\tau)-(1+4k^2)](\Delta y) \tag{3.22}$$

and

TABLE I. Cases of the generalized coupled quartic oscillator system, (2.1) and (2.2a), which have stability of type 1 with respect to two lines of initial value, and which are also integrable.

Case	$A:B$	$D:C:E$
I	1:1	1:1:1
II	1:1	1:3:1
III	1:4	1:6:16
IV	4:1	16:6:1
V	0:0	1:3:8
VI	0:0	8:3:1

$$\frac{d^2(\Delta x)}{d\tau^2}=[\tfrac{1}{2}\tfrac{3}{2}k^2\mathrm{sn}^2(\tau)-(1-\tfrac{5}{4}k^2)](\Delta x) \ . \tag{3.23}$$

We recognize Δy given by Eq. (3.22) to be the periodic Lamé function $\mathrm{Es}_2^1(\tau)$ as we have known from Eq. (3.13), for any value of k. On the other hand, the Δx given by Eq. (3.23) is not a periodic Lamé function for just any value of k, since the characteristic value for the Lamé equation of order $n=\frac{1}{2}$ is $\frac{1}{4}(1+k^2)$ (see the Appendix), and not $(1-\frac{5}{4}k^2)$. However, we now notice that $\frac{1}{4}(1+k^2)=1-\frac{5}{4}k^2$ for $k^2=\frac{1}{2}$. That is to say, if we let $A\to 0$, $B\to 0$ such that $A/B\to 1$, then the generalized coupled quartic oscillator system with $D:C:E=1:3:8$ has stability of type 1 with respect to two lines of initial values $(x_0,0,0,0)$ and $(0,y_0,0,0)$. Similarly we may deduce that the case with $D:C:E=8:3:1$ has stability of type 1 by setting $A\to 0$, $B\to 0$, $A/B=1$, $C/E=3$, and $C/D=\frac{3}{8}$ in Eqs. (3.8b) and (3.21). The two cases are listed as cases V and VI in Table I.

Cases V and VI turn out to be one of the new integrable cases discovered by Ramani, Dorizzi, and Grammaticos[7] using the Painlevé analysis. The second constant of motion was obtained by them to be (for case V, $D=1$)

$$\dot x^4+(24x^2y^2+4x^4)\dot x^2-16x^3y\dot x\dot y+4x^4\dot y^2 +4x^8+16x^6y^2+16x^4y^4=\mathrm{const} \ . \tag{3.24}$$

For completeness, we also give below the second constants of motion for cases I and II in Table I:

$$x\dot y-\dot x y=r^2\dot\theta=\mathrm{const},\quad \text{case I} \tag{3.25}$$

$$\dot x\dot y+Axy+4Dxy(x^2+y^2)=\mathrm{const},\quad \text{case II} \ . \tag{3.26}$$

Equation (3.26) was given (for $A=0$, $D=1$) by Yoshida.[26]

To sum up the result of this section, our consideration of stability of type 1 with respect to two lines of initial values for the generalized coupled quartic oscillator system has led us to the six known integrable cases. The usefulness of considering stability of type 1 becomes more apparent in the study of the generalized Hénon-Heiles system in the following section, when it leads to not only known integrable cases but also to six other cases.

IV. GENERALIZED HÉNON-HEILES SYSTEM

The Hamiltonian of the generalized Hénon-Heiles system is given by Eqs. (2.1) and (2.2b) and it contains four parameters A,B,C,D. The equations of motion are

$$\ddot x=-Ax+2Cxy \ , \tag{4.1a}$$

$$\ddot y=-By-Cx^2+Dy^2 \ . \tag{4.1b}$$

For small deviations Δx and Δy, their behavior is governed by

$$\begin{aligned}&\frac{d^2(\Delta x)}{dt^2}+(A+2Cy)(\Delta x)+2Cx(\Delta y)=0 \ ,\\ &\frac{d^2(\Delta y)}{dt^2}+2Cx(\Delta x)+(B-2Dy)(\Delta y)=0 \ .\end{aligned} \tag{4.2}$$

If $x(0)=\dot x(0)=0$, then $x(t)=0$ for all t, as can be seen from (4.1a), and the equation of motion for y is, from (4.1b),

$$\ddot y+By-Dy^2=0 \ . \tag{4.3}$$

We assume that $\dot y(0)=0$, and to have a bounded motion, we assume that the initial given energy $\mathcal{E}$ of the system is in the range

$$0\le\mathcal{E}\le\tfrac{1}{6}\frac{B^3}{D^2} \ , \tag{4.4}$$

or

$$c\le y(0)\le b \ , \tag{4.5}$$

where $a\ge b\ge c$ are the three real roots of the cubic equation for $y(0)$:

$$\tfrac{1}{2}By(0)^2-\frac{D}{3}y(0)^3=\mathcal{E} \ , \tag{4.6}$$

which is also the equation relating a given value of $y(0)$ to the energy $\mathcal{E}$ of the system, assuming $x(0)=\dot x(0)=\dot y(0)=0$. The solution of Eq. (4.3) for $y(0)=c$ is

$$y=c+(a-c)k^2\mathrm{sn}^2(\tau,k) \ , \tag{4.7}$$

where

$$\tau=\Omega t \ , \tag{4.8}$$

$$\Omega=\left[\frac{D}{6}(a-c)\right]^{1/2} \ , \tag{4.9}$$

and the modulus k of the elliptic function $\mathrm{sn}(\tau)$ is

$$k^2=\frac{b-c}{a-c} \ . \tag{4.10}$$

Thus y is a periodic function whose period in the rescaled time τ is $2K(k)$. Substituting Eqs. (4.7)–(4.10) into (4.2), we find that the behavior of any small deviations Δx and Δy from the line of initial values $(x(0),y(0),\dot x(0),\dot y(0))=(0,y_0,0,0)$ is governed by the following equations:

$$\frac{d^2(\Delta x)}{d\tau^2}=\left\{-\frac{12C}{D}k^2\mathrm{sn}^2(\tau)-\left[-\frac{4C}{D}(1+k^2)+4\left[\frac{A}{B}+\frac{C}{D}\right](1-k^2+k^4)^{1/2}\right]\right\}(\Delta x) \ , \tag{4.11a}$$

$$\frac{d^2(\Delta y)}{d\tau^2}=[12k^2\mathrm{sn}^2(\tau)-4(1+k^2)](\Delta y) \ . \tag{4.11b}$$

TABLE II. Eight cases of the generalized Hénon-Heiles system, (2.1) and (2.2b), which have stability of type 1 with respect to a line of initial values. Cases 1 and 6 are known to be integrable.

Case	A/B	C/D	Characteristic values	Periodic Lamé functions
1	$\frac{1}{16}$	$-\frac{1}{16}$	$a_{1/2}^{1/2}$	$\mathrm{Ec}_{1/2}^{1/2}(\tau)$ and $\mathrm{Es}_{1/2}^{1/2}(\tau)$
2	$\frac{1}{16}$	$-\frac{5}{16}$	$a_{3/2}^{1/2}$	$\mathrm{Ec}_{3/2}^{1/2}(\tau)$ and $\mathrm{Es}_{3/2}^{1/2}(\tau)$
3	$\frac{9}{16}$	$-\frac{5}{16}$	$a_{3/2}^{3/2}$	$\mathrm{Ec}_{3/2}^{3/2}(\tau)$ and $\mathrm{Es}_{3/2}^{3/2}(\tau)$
4	0	$-\frac{1}{2}$	a_2^0	$\mathrm{Ec}_2^0(\tau)$
5	1	$-\frac{1}{2}$	a_2^2	$\mathrm{Ec}_2^2(\tau)$
6	1	-1	b_3^2	$\mathrm{Es}_3^2(\tau)$
7	1	$-\frac{5}{2}$	b_5^2	$\mathrm{Es}_5^2(\tau)$
8	4	$-\frac{5}{2}$	b_5^4	$\mathrm{Es}_5^4(\tau)$

Equations (4.11) are again of the form of the Lamé equation (3.9). Requiring the system to have stability of type 1 with respect to the chosen line of initial values again implies requiring Δx and Δy to be periodic Lamé functions for all allowed values of y_0 or k.

From the Appendix, we see that for Eq. (4.11b), Δy is given by the periodic Lamé polynomial $\mathrm{Es}_3^2(\tau)$ for all allowed values of y_0 or k with the characteristic value $b_3^2=4(1+k^2)$, for any values of A,B,C,D in the Hamiltonian. On the other hand, for Δx in Eq. (4.11a) to be periodic and thus to be a periodic Lamé function, the quantity in the large square brackets in (4.11a) should be equated with one of the characteristic values listed in the Appendix, and then the value of C/D so obtained must agree with the order n of the characteristic value and function given by $n(n+1)=-12C/D$. When these are satisfied, Δx will be given by the periodic Lamé function, and the generalized Hénon-Heiles system has stability of type 1 with respect to the line of initial values $(x(0),y(0),\dot{x}(0),\dot{y}(0))=(0,y_0,0,0)$ for all allowed values of y_0.

The result of the above analysis gives eight possibilities for the system to have stability of type 1 with respect to the chosen line. The values of A/B and C/D for these eight possibilities are given in the second and third columns of Cases 1–8 listed in Table II, where the last two columns of the table give the corresponding characteristic values and functions for Δx.

Case 1 in Table II was shown by Chang, Tabor, and Weiss[5] to possess the Painlevé property and was conjectured to be integrable. The second integral of motion was subsequently obtained by Grammaticos, Dorizzi, and Padjen[6] to be

$$3\dot{x}^4+6(A+2Cy)x^2\dot{x}^2-4Cx^3\dot{x}\dot{y}-4Cx^4(Ay+Cy^2)+3A^2x^4-\tfrac{2}{3}C^2x^6=\text{const} . \quad (4.12)$$

Case 6 is a known integrable case for which the Hamiltonian can be shown to be separable by rotating the x-y coordinates by 45°. The second integral of motion for this case is

$$\dot{x}\dot{y}+Axy+\tfrac{1}{3}C(x^3+3xy^2)=\text{const} . \quad (4.13)$$

Cases 7 and 8 have recently been proved by Yoshida (see *Notes added*) to be nonintegrable. The remaining four cases (cases 2–5) are not known to be integrable or nonintegrable. It would be interesting if one could understand why these six cases have such characteristic like stability of type 1 as some integral cases have.

We note that case 5 was a case found by Ito[21] which satisfied his condition for the system to have an entire integral which is functionally independent of the Hamiltonian of the system, but he was not able to prove the integrability or nonintegrability of this case.

A notable case that is not one of this list of eight cases is the case A/B=any number, $C/D=-\frac{1}{6}$, which was conjectured to be integrable[5] and for which the second integral of motion was known. The second integral is

$$(4A-B)(\dot{x}^2+Ax^2)+4C\dot{x}(x\dot{y}-\dot{x}y)+4CAx^2y+4C^2x^2y^2+C^2x^4=\text{const} , \quad (4.14)$$

which was given (for $C=-1$) by Greene.[27] In particular, a special case of this for which $A/B=\frac{1}{4}$ was known

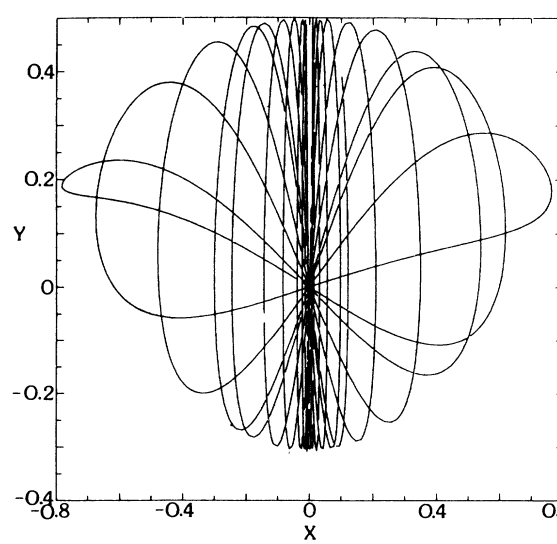

FIG. 1. The x-y trajectory of the generalized Hénon-Heiles system, (2.1) and (2.2b), for $A=1$, $B=4$, $C=-1$, and $D=6$, and for the initial values $x(0)=0.01$, $y(0)=0.5$, $\dot{x}(0)=\dot{y}(0)=0$.

to be separable in the parabolic coordinates.[17] The reason this integrable case was missed out by our stability analysis is made immediately clear in Fig. 1 which shows the x-y trajectory of the motion for case $A=1$, $B=4$, $C=-1$, $D=6$ for the initial values $(x(0),y(0),\dot{x}(0),\dot{y}(0))=(0.01,0.5,0,0)$. It is seen that even though the trajectory is clearly regular, it shows an unstable behavior with respect to any small deviations Δx from the very line of initial values $(0,y_0,0,0)$ which we have used to determine the first eight cases in Table II. In fact, it can be shown analytically that any small deviation from the line of initial values $(0,y_0,0,0)$ gives an unstable behavior for this case $(A/B=\frac{1}{4}, C/D=-\frac{1}{6})$. This instability can occur because the gradients of the first and second integrals become linearly dependent on the straight line $(x=\dot{x}=0)$, and the case is thus not covered by Poincaré's theorem[28,29] on the nonexistence of an exponentially unstable solution in an integrable Hamiltonian system with two degrees of freedom. This rare occurrence does happen here, as can be verified from Eqs. (4.14) and (2.2b). This example clearly shows that the method of stability of type 1 may not lead to all possible integrable cases, and we certainly do not claim that it would.

V. SUMMARY

We have introduced the concept of stability of type 1 and applied it to (1) the generalized coupled quartic oscillator system, and (2) the generalized Hénon-Heiles system. By requiring the systems to have stability of type 1 with respect to two specific lines of initial values for (1), and with respect to one specific line of initial values for (2), we have been led to all six previously known integrable cases for (1), and to two of the three previously known integrable cases for (2). In addition, we have six other cases for (2) which have stability of type 1 with respect to the same line of initial values, two of which turn out to be nonintegrable. One of these six cases (case 5) was obtained previously by Ito[21] to be one which could have an entire integral which is functionally independent of the Hamiltonian of the system. We believe that the method of stability of type 1 should be useful for studying other Hamiltonian systems in two and higher dimensions, for it has led to a "family" of cases which includes the integrable cases.

Notes added. (1) A preliminary report of the main result of this paper has been published.[30] Note that in Eq. (1) of that Letter, C should be replaced by $2C$. Cases V and VI of Table I reported in the present paper were not

TABLE III. List of all the characteristic values of the Lamé equation (3.9) that can be expressed analytically in the form given by (3.10), and the corresponding periodic characteristic functions or periodic Lamé functions. We use the notations used by Ince (Ref. 24).

n	Characteristic values	Periodic Lamé functions
$\frac{1}{2}$	$a_{1/2}^{1/2}=\frac{1}{4}(1+k^2)$	$\mathrm{Ec}_{1/2}^{1/2}(\tau)=\dfrac{\mathrm{cn}(\frac{1}{2}\tau)\mathrm{dn}(\frac{1}{2}\tau)}{[1-k^2\mathrm{sn}^4(\frac{1}{2}\tau)]^{1/2}}$
		$\mathrm{Es}_{1/2}^{1/2}(\tau)=\dfrac{\mathrm{sn}(\frac{1}{2}\tau)}{[1-k^2\mathrm{sn}^4(\frac{1}{2}\tau)]^{1/2}}$
1	$a_1^0=k^2$	$\mathrm{Ec}_1^0(\tau)=\mathrm{dn}(\tau)$
	$a_1^1=1$	$\mathrm{Ec}_1^1(\tau)=\mathrm{cn}(\tau)$
	$b_1^1=1+k^2$	$\mathrm{Es}_1^1(\tau)=\mathrm{sn}(\tau)$
$\frac{3}{2}$	$a_{3/2}^{1/2}=\frac{5}{4}(1+k^2)-(1-k^2+k^4)^{1/2}$	$\mathrm{Ec}_{3/2}^{1/2}(\tau)=\dfrac{\mathrm{cn}(\frac{1}{2}\tau)\mathrm{dn}(\frac{1}{2}\tau)\{\mathrm{dn}(\tau)-[1-k^2-(1-k^2+k^4)^{1/2}]\mathrm{cn}(\tau)\}}{[1-k^2\mathrm{sn}^4(\frac{1}{2}\tau)]^{1/2}}$
		$\mathrm{Es}_{3/2}^{1/2}(\tau)=\dfrac{\mathrm{sn}(\frac{1}{2}\tau)\{\mathrm{dn}(\tau)+[1-k^2-(1-k^2+k^4)^{1/2}]\mathrm{cn}(\tau)\}}{[1-k^2\mathrm{sn}^4(\frac{1}{2}\tau)]^{1/2}}$
	$a_{3/2}^{3/2}=\frac{5}{4}(1+k^2)+(1-k^2+k^4)^{1/2}$	$\mathrm{Ec}_{3/2}^{3/2}(\tau)=\dfrac{\mathrm{cn}(\frac{1}{2}\tau)\mathrm{dn}(\frac{1}{2}\tau)\{\mathrm{dn}(\tau)-[1-k^2+(1-k^2+k^4)^{1/2}]\mathrm{cn}(\tau)\}}{[1-k^2\mathrm{sn}^4(\frac{1}{2}\tau)]^{1/2}}$
		$\mathrm{Es}_{3/2}^{3/2}(\tau)=\dfrac{\mathrm{sn}(\frac{1}{2}\tau)\{\mathrm{dn}(\tau)+[1-k^2+(1-k^2+k^4)^{1/2}]\mathrm{cn}(\tau)\}}{[1-k^2\mathrm{sn}^4(\frac{1}{2}\tau)]^{1/2}}$
2	$a_2^0=2(1+k^2)-2(1-k^2+k^4)^{1/2}$	$\mathrm{Ec}_2^0(\tau)=1-[1+k^2-(1-k^2+k^4)^{1/2}]\mathrm{sn}^2(\tau)$
	$a_2^1=1+k^2$	$\mathrm{Ec}_2^1(\tau)=\mathrm{cn}(\tau)\mathrm{dn}(\tau)$
	$b_2^1=1+4k^2$	$\mathrm{Es}_2^1(\tau)=\mathrm{sn}(\tau)\mathrm{dn}(\tau)$
	$b_2^2=4+k^2$	$\mathrm{Es}_2^2(\tau)=\mathrm{sn}(\tau)\mathrm{cn}(\tau)$
	$a_2^2=2(1+k^2)+2(1-k^2+k^4)^{1/2}$	$\mathrm{Ec}_2^2(\tau)=1-[1+k^2+(1-k^2+k^4)^{1/2}]\mathrm{sn}^2(\tau)$
3	$b_3^2=4(1+k^2)$	$\mathrm{Es}_3^2(\tau)=\mathrm{sn}(\tau)\mathrm{cn}(\tau)\mathrm{dn}(\tau)$
5	$b_5^2=10(1+k^2)-6(1-k^2+k^4)^{1/2}$	$\mathrm{Es}_5^2(\tau)=\mathrm{cn}(\tau)\mathrm{dn}(\tau)\{\mathrm{sn}(\tau)-[1+k^2-(1-k^2+k^4)^{1/2}]\mathrm{sn}^3(\tau)\}$
	$b_5^4=10(1+k^2)+6(1-k^2+k^4)^{1/2}$	$\mathrm{Es}_5^4(\tau)=\mathrm{cn}(\tau)\mathrm{dn}(\tau)\{\mathrm{sn}(\tau)-[1+k^2+(1-k^2+k^4)^{1/2}]\mathrm{sn}^3(\tau)\}$

given in the earlier report. (2) Two recent papers by Yoshida[31] gave a nonintegrability criterion for a two-dimensional Hamiltonian system which rules out the integrability of cases 7 and 8 of Table II.

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APPENDIX

In this appendix, we refer the reader to Table III.

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